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# FAST MEAN-REVERSION ASYMPTOTICS FOR LARGE PORTFOLIOS OF STOCHASTIC VOLATILITY MODELS

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# Fast mean-reversion asymptotics for large portfolios of stochastic volatility models

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**Abstract**

We consider a large portfolio limit where the asset prices evolve according certain stochastic volatility models with default upon hitting a lower barrier. When the asset prices and the volatilities are correlated via systemic Brownian Motions, that limit exist and it is described by a SPDE on the positive half-space with Dirichlet boundary conditions which has been studied in [12]. We study the convergence of the total mass of a solution to this stochastic initial-boundary value problem when the mean-reversion coefficients of the volatilities are multiples of a parameter that tends to infinity. When the volatilities of the volatilities are multiples of the square root of the same parameter, the convergence is extremely weak. On the other hand, when the volatilities of the volatilities are independent of this exploding parameter, the volatilities converge to their means and we can have much better approximations. Our aim is to use such approximations to improve the accuracy of certain risk-management methods in markets where fast volatility mean-reversion is observed.

## 1 Introduction

An interesting way to handle the complexity of the calibration of stochastic volatility models is developed in [9]. The prices of several vanilla and exotic options under a stochastic volatility model are written as series of negative powers of the mean-reversion coefficient of the volatility process. Since the value of the mean-reversion coefficient is generally observed to be large, by keeping the first two or three terms of these series we obtain good approximations for the option prices, which are more accurate than the corresponding Black-Scholes prices - obtained by keeping only the first term - while their computation is much simpler than the computation of the exact prices under the full stochastic volatility model.

In this paper our aim is to follow the ideas of [9] but instead of option prices look at the systemic risk of such models in the large portfolio setting. Stochastic volatility models in the large portfolio setting were first introduced in [12] where a two-dimensional SPDE was derived for the density of asset prices and volatility as the large portfolio limit. The existence of solutions to the SPDE was established but the uniqueness of solutions to the SPDE remains an open question in the CIR volatility case. This is a significant issue

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when constructing numerical solutions to this SPDE. Ideally, we would like to construct an approximate model, which goes beyond the constant volatility model studied in [6], but which is also easier to handle than the full two-dimensional models. Unfortunately, we are only able to show convergence to a constant volatility model as a factor of the mean-reversion coefficients tends to infinity, and the existence of a first order correction provided that the volatilities of the volatilities are independent of the exploding factor (small vol-of-vol case). Moreover, the errors tend to zero only in a weak sense, and we are also unable to determine the correction explicitly, since even though it solves an SPDE on the positive half-line, we are not able to derive a boundary condition at zero. However, in the small vol-of-vol case, we can estimate the rate of convergence to a constant volatility model, and this is the best possible result we can have at this stage.

As discussed in [12], the main applications of large portfolio modelling arise in risk management and in the pricing of derivatives like CDO tranches. In this paper we will focus on the applications in risk management, where we need to estimate the probability that the total loss $L_t$ within the portfolio at some time $t > 0$ exceeds a certain proportion. Following the notation and the results from [12], we would have that $L_t = 1 - \mathbb{P}\left(X_t^1 > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)$, where $X_\cdot^i$ stands for the $i$-th logarithmically scaled price process which satisfies the system of SDEs

$$\begin{aligned} dX_t^i &= \left(r_i - \frac{h^2(\sigma_t^i)}{2}\right)dt + h(\sigma_t^i)\left(\sqrt{1-\rho_{1,i}^2}dW_t^i + \rho_{1,i}dW_t^0\right), \quad 0 \le t \le T_i \\ d\sigma_t^i &= k_i(\theta_i - \sigma_t^i)dt + \xi_i\sqrt{\sigma_t^i}\left(\sqrt{1-\rho_{2,i}^2}dB_t^i + \rho_{2,i}dB_t^0\right), \quad t \ge 0 \\ X_t^i &= 0, \ t > T_i \\ (X_0^i,\, \sigma_0^i) &= (x^i, \sigma^i), \end{aligned} \tag{1.1}$$

for all $i \in \{1, 2, ...\}$, with $T_i$ being the first time $X_\cdot^i$ hits 0, $C_i = (k_1,\, \theta_1,\, \xi_1,\, r_1,\, \rho_{1,1},\, \rho_{2,1})$ for $i = 1, 2, ...$ being i.i.d random vectors such that $k_i\theta_i > \frac{3\xi_i^2}{4}$ a.s for every $i \in \{1, 2, ...\}$, $W_\cdot^1$, $B_\cdot^1$, $W_\cdot^2$, $B_\cdot^2$, ... being pairwise independent standard Brownian Motions representing idiosyncratic factors that affect each asset's price, $W_\cdot^0$, $B_\cdot^0$ being two (possibly correlated) standard Brownian Motions describing the interdependence among the asset prices, and $(x^i, \sigma^i)$ for $i = 1, 2, ...$ being random vectors with positive coordinates which are pairwise independent given $\mathcal{G}$. However, here we will consider a more general model, in which the $i$-th logarithmically scaled asset price process $X_\cdot^i$ satisfies the system

$$\begin{aligned} dX_t^i &= \left(r_i - \frac{h^2(\sigma_t^i)}{2}\right)dt + h(\sigma_t^i)\left(\sqrt{1-\rho_{1,i}^2}dW_t^i + \rho_{1,i}dW_t^0\right), \quad 0 \le t \le T_i \\ d\sigma_t^i &= k_i(\theta_i - \sigma_t^i)dt + \xi_i g\left(\sigma_t^i\right)\left(\sqrt{1-\rho_{2,i}^2}dB_t^i + \rho_{2,i}dB_t^0\right), \quad t \ge 0 \\ X_t^i &= 0, \ t > T_i \\ (X_0^i,\, \sigma_0^i) &= (x^i, \sigma^i), \end{aligned} \tag{1.2}$$

for all $i \in \{1, 2, ...\}$, where the function $g$ is chosen such that the volatility processes possess certain crucial properties. Thus, under the above notation, our aim is to estimate probabilities of the form:

$$\mathbb{P}\left(L_t \in (1-b,\, 1-a)\right) = \mathbb{P}\left(\mathbb{P}\left(X_t^1 > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right) \in (a,\, b)\right) \tag{1.3}$$

for some $0 \le a < b \le 1$.

The natural way to adapt the ideas from [9] to our setting is to set $k_i = \frac{\kappa_i}{\epsilon}$ and $\xi_i = \frac{v_i}{\sqrt{\epsilon}}$ for all $i = 1, 2, ...$, with $\{\kappa_i : i \in \{1, 2, ...\}\}$ and $\{v_i : i \in \{1, 2, ...\}\}$ being sequences of non-negative i.i.d random variables that do not depend on $\epsilon$, and then try to approximate the quantity in (1.3) by something more accurate than its limit as $\epsilon \to 0^+$. However, we will see that in this fast mean-reversion - large vol-of-vol setting, the convergence of the system as $\epsilon \to 0^+$ is very weak, which does not allow us to hope for such an approximation. Moreover, this convergence can only be obtained under the unrealistic assumption that the market noises $W^0_\cdot$ and $B^0_\cdot$ are uncorrelated. For this reason, we will also consider the fast mean-reversion - small vol-of-vol setting, where again we have $k_i = \frac{\kappa_i}{\epsilon}$ for all $i = 1, 2, ...$, but this time $\{\xi_i : i \in \{1, 2, ...\}\}$ is a sequence of non-negative i.i.d random variables which does not depend on $\epsilon$. We will show in section 6 that in this small vol-of-vol setting we can also estimate the rate of convergence as $\epsilon \to 0^+$, even for correlated market noises, provided that a certain regularity condition is satisfied at both $a$ and $b$.

## 2 Fast mean-reversion - large vol-of-vol: A first approach

We begin with the study of the fast mean-reversion - large vol-of-vol setting, for which we need to assume that $W^0_\cdot$ and $B^0_\cdot$ are uncorrelated. It has been proven in [12] that

$$\mathbb{P}\left(X^1_t > 0 \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right) = \mathbb{E}\left[\int_0^{+\infty}\int_0^{+\infty} u_{C_1}(t,\, x,\, y)\, dxdy \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right] \tag{2.1}$$

where $u_{C_1}$ is a regular solution to the SPDE

$$\begin{aligned} u_{C_1}(t,\, x,\, y) \;&=\; U_0(x,\, y \,|\, G) - r_1 \int_0^t (u_{C_1}(s,\, x,\, y))_x\, ds \\ &\quad + \frac{1}{2}\int_0^t h^2(y)\,(u_{C_1}(s,\, x,\, y))_x\, ds - k_1\theta_1 \int_0^t (u_{C_1}(s,\, x,\, y))_y\, ds \\ &\quad + k_1 \int_0^t (y u_{C_1}(s,\, x,\, y))_y\, ds + \frac{1}{2}\int_0^t h^2(y)\,(u_{C_1}(s,\, x,\, y))_{xx}\, ds \\ &\quad + \xi_1\rho_3\rho_{1,1}\rho_{2,1} \int_0^t (h\,(y)\, g\,(y)\, u_{C_1}(s,\, x,\, y))_{xy}\, ds \\ &\quad + \frac{\xi_1^2}{2}\int_0^t \left(g^2\,(y)\, u_{C_1}(s,\, x,\, y)\right)_{yy} ds \\ &\quad - \rho_{1,1}\int_0^t h(y)\,(u_{C_1}(s,\, x,\, y))_x\, dW^0_s \\ &\quad - \xi_1\rho_{2,1}\int_0^t (g\,(y)\, u_{C_1}(s,\, x,\, y))_y\, dB^0_s, \end{aligned} \tag{2.2}$$

in the half-space $\mathbb{R}^+ \times \mathbb{R}$, and expectation in (2.1) is taken to average over all the possible values of the coefficient vector $C_1$. Substituting from $k_1 = \frac{\kappa_1}{\epsilon}$ and $\xi_1 = \frac{v_1}{\sqrt{\epsilon}}$ and writing $C^\epsilon_1$ for $C_1$ to mention the dependence on $\epsilon$, the above SPDE can be written as

$$\begin{aligned} u_{C^\epsilon_1}(t,\, x,\, y) \;&=\; U_0(x,\, y \,|\, G) - r_1 \int_0^t \left(u_{C^\epsilon_1}(s,\, x,\, y)\right)_x ds \\ &\quad + \frac{1}{2}\int_0^t h^2(y)\left(u_{C^\epsilon_1}(s,\, x,\, y)\right)_x ds - \frac{\kappa_1\theta_1}{\epsilon}\int_0^t \left(u_{C^\epsilon_1}(s,\, x,\, y)\right)_y ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa_1}{\epsilon}\int_0^t \left(y u_{C_1^\epsilon}(s,\,x,\,y)\right)_y ds+\frac{1}{2}\int_0^t h^2(y)\left(u_{C_1^\epsilon}(s,\,x,\,y)\right)_{xx} ds\\
&+\frac{v_1}{\sqrt{\epsilon}}\rho_3\rho_{1,1}\rho_{2,1}\int_0^t \left(h\left(y\right)g\left(y\right)u_{C_1^\epsilon}(s,\,x,\,y)\right)_{xy} ds\\
&+\frac{v_1^2}{2\epsilon}\int_0^t \left(g^2\left(y\right)u_{C_1^\epsilon}(s,\,x,\,y)\right)_{yy} ds\\
&-\rho_{1,1}\int_0^t h(y)\left(u_{C_1^\epsilon}(s,\,x,\,y)\right)_x dW_s^0\\
&-\frac{v_1}{\sqrt{\epsilon}}\rho_{2,1}\int_0^t \left(g\left(y\right)u_{C_1^\epsilon}(s,\,x,\,y)\right)_y dB_s^0, \qquad (2.3)
\end{aligned}
$$

A reasonable approach to our problem is to try to expand $u_{C_1}^{\epsilon}$ as a series of natural powers of $\epsilon^p$ for some $p > 0$, with the coefficients being random functions of $t$, $x$ and $y$, substitute in (2.3), and solve the stochastic equations arising by equating coefficients. However, due to the presence of the stochastic integral with respect to $B^0_\cdot$, we obtain two equations whenever we equate the coefficients of two terms of the form $\epsilon^{np}$, and this renders the system of stochastic equations arising this way unsolvable. This difficulty we face has also a second explanation: In the CIR case (model 1.1), $u_{C_1^\epsilon}$ has been computed explicitly in [12] and it is equal to

$$
p_t^\epsilon\left(y|B^0_\cdot,\mathcal{G}\right)\mathbb{E}\left[u\left(t,x,W^0_\cdot,\mathcal{G},C_1^\epsilon,h\left(\sigma^{1,\epsilon}_\cdot\right)\right)|W^0_\cdot,\sigma_t^{1,\epsilon}=y,B^0_\cdot,C_1^\epsilon,\mathcal{G}\right]
$$

where we write $\sigma^{1,\epsilon}_\cdot$ for $\sigma^1_\cdot$ to mention again the dependence on $\epsilon$, and $p_t^\epsilon$ for the density of the volatility process when the path of $B^0_\cdot$ is given. Under appropriate restrictions on the function $g$, we can extend the above formula to the more general case (model (1.2)). Obviously, $p_t^\epsilon$ contains information for the path of $B^0_\cdot$ and, intuitively, this means that we need pathwise convergence of $\sigma^{1,\epsilon}_\cdot$ as $\epsilon \to 0^+$ to have convergence of this density. On the other hand, the volatility processes we consider here converge weakly when we take the mean-reversion coefficient tending to infinity. However, since the above problem is mainly caused by $p_t^\epsilon$, we hope that we may be able to obtain some good results by controlling appropriately $p_t^\epsilon$ and by trying to approximate the second factor of $u_{C_1^\epsilon}$, i.e the term

$$
\mathbb{E}\left[u\left(t,x,W^0_\cdot,\mathcal{G},C_1^\epsilon,h\left(\sigma^{1,\epsilon}_\cdot\right)\right)|W^0_\cdot,\sigma_t^{1,\epsilon}=y,B^0_\cdot,C_1^\epsilon,\mathcal{G}\right].
$$

We observe now that since we are interested in the probability of an event concerning the loss process, which depends on the conditional density $p_t^\epsilon$ and the above conditional expectation, we can replace $\left(W^1_\cdot,\,W^0_\cdot,\,B^1_\cdot,\,B^0_\cdot,\,C_1^\epsilon\right)$ by anything having the same law for each $\epsilon > 0$. We look thus at the SDE satisfied by the process $\sigma^{1,\epsilon}_\cdot$, i.e

$$
\sigma_t^{1,\epsilon}=\sigma_0^{1,\epsilon}+\frac{\kappa_1}{\epsilon}\int_0^t(\theta_1-\sigma_s^{1,\epsilon})ds+\frac{v_1}{\sqrt{\epsilon}}\int_0^t g\left(\sigma_s^{1,\epsilon}\right)d\left(\sqrt{1-\rho_{2,1}^2}B_s^1+\rho_{2,1}B_s^0\right). \qquad (2.4)
$$

and we observe that if we substitute $t = \epsilon t'$ and $s = \epsilon s'$ for $0 \leq s' \leq t'$, and then we replace $\left(W^1_\cdot,\,W^0_\cdot,\,B^1_\cdot,\,B^0_\cdot,\,C_1^\epsilon\right)$ by $\left(W^1_\cdot,\,W^0_\cdot,\,\sqrt{\epsilon}B^1_{\frac{\cdot}{\epsilon}},\,\sqrt{\epsilon}B^0_{\frac{\cdot}{\epsilon}},\,C_1^\epsilon\right)$ which has the same law, the above SDE becomes

$$
\sigma_{\epsilon t'}^{1,\epsilon}=\sigma_0^{1,\epsilon}+\kappa_1\int_0^{t'}(\theta_1-\sigma_{\epsilon s'}^{1,\epsilon})ds'+v_1\int_0^{t'} g\left(\sigma_{\epsilon s'}^{1,\epsilon}\right)d\left(\sqrt{1-\rho_{2,1}^2}B_{s'}^1+\rho_{2,1}B_{s'}^0\right). \qquad (2.5)
$$

This shows that $\sigma_{\epsilon\cdot}^{1,\epsilon}$ can be replaced by the volatility process of our model when the coefficient vector $C_1$ is replaced by $C_1' = (\kappa_1,\, \theta_1,\, v_1,\, r_1,\, \rho_{1,1},\, \rho_{2,1})$, which is just $\sigma_{\cdot}^{1,1}$ (the volatility process when $\epsilon = 1$). Thus, we can replace $\sigma_t^{1,\epsilon}$ by $\sigma_{\frac{t}{\epsilon}}^{1,1}$ for all $t \geq 0$, which allows us to replace

$$\mathbb{E}\left[u\left(t, x, W_{\cdot}^0, \mathcal{G}, C_1^{\epsilon}, h\left(\sigma_{\cdot}^{1,\epsilon}\right)\right) |W_{\cdot}^0, \sigma_t^{1,\epsilon} = y, B_{\cdot}^0, C_1^{\epsilon}, \mathcal{G}\right]$$

by the conditional expectation

$$\mathbb{E}\left[u\left(t, x, W_{\cdot}^0, \mathcal{G}, C_1', h\left(\sigma_{\frac{\cdot}{\epsilon}}^{1,1}\right)\right) |W_{\cdot}^0, \sigma_{\frac{t}{\epsilon}}^{1,1} = y, B_{\cdot}^0, C_1', \mathcal{G}\right].$$

The above quantity is what we need to approximate now. Of course, a first step is to show the convergence of that conditional expectation as $\epsilon \to 0^+$, which motivates us to look for some kind of convergence for the random function $u^{\epsilon}(t,x) := u\left(t, x, W_{\cdot}^0, \mathcal{G}, C_1', h\left(\sigma_{\frac{\cdot}{\epsilon}}^{1,1}\right)\right)$ as $\epsilon \longrightarrow 0^+$, when the volatility path $\sigma_{\cdot}^{1,1}$ and the coefficient vector $C_1'$ are given.

The approach explained above is the subject of the next section, but we will also see a different approach to the same problem in section 4. The two approaches are going to give different limits, and thus it will become clear that the convergence of our system is so weak that no good approximations should be expected. In both approaches, we need to assume that our function $g$ is chosen such that every pair of volatility processes has a nice ergodic behaviour. This property is defined below

**Definition 2.1** (**Positive recurrence property).** We fix the distribution from which each $C_i$ is chosen, and we denote by $\mathcal{C}$ the $\sigma$-algebra generated by all the $C_i$s. Then, we say that $g$ has the strong positive recurrence property when the two-dimensional process $\left(\sigma_{\cdot}^{i,1},\, \sigma_{\cdot}^{j,1}\right)$ is a positive recurrent diffusion for any two $i, j \in \mathbb{N}$. This means that there exists a two-dimensional random variable $\left(\sigma^{i,j,1,*},\, \sigma^{i,j,2,*}\right)$ whose distribution is stationary for $\left(\sigma_{\cdot}^{i,1},\, \sigma_{\cdot}^{j,1}\right)$, and whenever $\mathbb{E}\left[\left|F\left(\sigma^{i,j,1,*},\, \sigma^{i,j,2,*}\right)\right| |\mathcal{C}\right]$ exists and is finite for some function $F : \mathbb{R}^2 \to \mathbb{R}$, we also have:

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T F\left(\sigma_s^{i,1},\, \sigma_s^{j,1}\right) ds = \mathbb{E}\left[F\left(\sigma^{i,j,1,*},\, \sigma^{i,j,2,*}\right) |\mathcal{C}\right] \tag{2.6}$$

$\mathbb{P}$-almost surely.

*Remark* 2.2. By a change of variables, we can easily verify that when we have (2.6), we also have

$$\lim_{\epsilon\to0^+} \frac{1}{t}\int_0^t F\left(\sigma_{\frac{s}{\epsilon}}^{i,1},\, \sigma_{\frac{s}{\epsilon}}^{j,1}\right) ds = \mathbb{E}\left[F\left(\sigma^{i,j,1,*},\, \sigma^{i,j,2,*}\right) |\mathcal{C}\right] \tag{2.7}$$

$\mathbb{P}$-almost surely, for any $t > 0$, and this shows why we have chosen to change the timescale of our volatility processes by replacing $\left(B_{\cdot}^1,\, B_{\cdot}^0,\, C_1^{\epsilon}\right)$ by $\left(\sqrt{\epsilon}B_{\frac{\cdot}{\epsilon}}^1,\, \sqrt{\epsilon}B_{\frac{\cdot}{\epsilon}}^0,\, C_1^{\epsilon}\right)$

Before proceeding to the convergence result we have for $u^{\epsilon}$ as $\epsilon \to 0^+$, we will mention two Theorems which give us a few classes of models for which the positive recurrence property is satisfied. The first theorem shows that for the Ornstein-Uhlenbeck model ($g(x) = 1$ for all $x \in \mathbb{R}$) we have always the positive recurrence property. The second theorem shows that for the CIR model ($g(x) = \sqrt{|x|}$ for all $x \in \mathbb{R}$) we have the positive recurrence property provided that the random coefficients of the volatilities satisfy certain conditions.

**Theorem 2.3.** *Suppose that $g$ is a differentiable function, bounded from below by some $c_g > 0$. Suppose also that $g'(x)\kappa_i(\theta_i - x) < \kappa_i g(x) + \frac{v_i}{2}g''(x)g^2(x)$ for all $x \in \mathbb{R}$ and $i \in \mathbb{N}$, for all possible values of $C_i$. Then $g$ has the positive recurrence property.*

**Theorem 2.4.** *Suppose that $g(x) = \sqrt{|x|}\tilde{g}(x)$, where the function $\tilde{g}$ is a continuously differentiable, strictly positive and increasing function taking values in $[c_g, 1]$ for some $c_g > 0$. Then, there exists an $\eta > 0$ such that $g$ has the positive recurrence property when $\|C_i - C_j\|_{L^\infty(\mathbb{R}^6)} < \eta$ and $\frac{\kappa_i}{v_j^2} > \frac{1}{4} + \frac{1}{\sqrt{2}}$ for all $i, j \in \mathbb{N}$, $\mathbb{P}$ - almost surely.*

The proofs of the above two theorems can be found in the Appendix.

# 3 Large mean reversion - large vol-of-vol: Weak convergence of $u^\epsilon$

In this section, we fix the volatility path $\sigma_\cdot^{1,1}$ and the coefficient vector $C_1'$, which means that all expectations are taken given $\sigma_\cdot^{1,1}$ and $C_1'$. We will write $\mathbb{E}_{\sigma,\mathcal{C}}$ do denote these expectations, and $L^2_{\sigma,\mathcal{C}}$ to denote the corresponding $L^2$ norms. Given the pair $\left(\sigma_\cdot^{1,1}, C_1'\right)$, we prove that $u^\epsilon$ does indeed converge as $\epsilon \to 0^+$. We can only prove weak convergence, but we are able to characterize the limit provided that the function $h$ satisfies a few boundedness conditions.

We start by recalling Theorem 4.1 from [12], according to which, $u^\epsilon$ is the unique solution to the following SPDE:

$$\begin{aligned} u^\epsilon(t, x) &= u_0(x) - \int_0^t \left( r - \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} \right) u^\epsilon_x(s, x)ds \\ &\quad + \int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} u^\epsilon_{xx}(s, x)ds - \rho_{1,1}\int_0^t h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right) u^\epsilon_x(s, x)dW^0_s \end{aligned} \tag{3.1}$$

for which we have also the identity

$$\|u^\epsilon(t, \cdot)\|^2_{L^2(\mathbb{R}^+)} + \left(1 - \rho^2_{1,1}\right)\int_0^t h^2\left(\sigma^{1,1}_{\frac{t}{\epsilon}}\right) \|u^\epsilon_x(s, \cdot)\|^2_{L^2(\mathbb{R}^+)} ds = \|u_0\|^2_{L^2(\mathbb{R}^+)}. \tag{3.2}$$

where $u_0$ stands for the common probability density of the asset prices at $t = 0$, given the information of the $\sigma$-algebra $\mathcal{G}$. The last identity shows that the $L^2(\mathbb{R}^+)$ norms of the solutions $u^\epsilon$, and their $L^2([0, T] \times \mathbb{R}^+)$ norms as well (for any $T > 0$), are all uniformly bounded by a random variable which has a finite $L^2_{\sigma,\mathcal{C}}(\Omega)$ norm (the assumptions made in [12] are also needed for this). It follows that in a subsequence of any given sequence of values of $\epsilon$ which tends to zero, we have weak convergence to some element $u$, and we can have this both in $L^2_{\sigma,\mathcal{C}}([0, T] \times \mathbb{R}^+ \times \Omega)$ and $\mathbb{P}$-almost surely in $L^2([0, T] \times \mathbb{R}^+)$. The characterization of the weak limits $u$ is given in the following theorem.

**Theorem 3.1.** *Suppose that $g$ has the positive recurrence property and that for some $C > 0$ we have $|h(x)| \leq C$ for all $x \geq 0$. Any weak limit $u$ of $u^\epsilon$ in $L^2_{\sigma,\mathcal{C}}([0, T] \times \mathbb{R}^+ \times \Omega)$ solves the following SPDE*

$$
\begin{aligned}
u(t,\,x) &= u_0(x) - \left(r - \frac{\sigma_{1,1}^2}{2}\right)\int_0^t u_x(s,\,x)ds \\
&\quad + \frac{\sigma_{1,1}^2}{2}\int_0^t u_{xx}(s,\,x)ds - \rho_{1,1}\sigma_{2,1}\int_0^t u_x(s,\,x)dW_s^0
\end{aligned} \tag{3.3}
$$

*for* $\sigma_{1,1} = \sqrt{\mathbb{E}\left[h^2\left(\sigma^{1,*}\right)\,|\,\mathcal{C}\right]}$ *and* $\sigma_{2,1} = \mathbb{E}\left[h\left(\sigma^{1,*}\right)\,|\,\mathcal{C}\right]$*, where* $\sigma^{1,*}$ *is a random variable following the stationary distribution of our volatility process* $\sigma^{1,1}_{\cdot}$*. If* $h$ *is bounded from below by a positive constant* $c > 0$*, the same weak convergence holds also in* $H_0^1(\mathbb{R}^+) \times L^2_{\sigma,\mathcal{C}}(\Omega \times [0,\,T])$*, and* $u$ *is then the unique solution to* (3.3) *in that space. The last means that there is a unique subsequential weak limit, and thus we have weak convergence as* $\epsilon \to 0^+$*.*

*Proof.* Let $\mathbb{V}$ be the set of $W^0_{\cdot}$-adapted, square-integrable semimartingales on $[0,\,T]$. This means that for any $\{V_t : 0 \le t \le T\} \in \mathbb{V}$, there exist two $W^0_{\cdot}$-adapted and square-integrable stochastic processes $\{v_{1,t} : 0 \le t \le T\}$ and $\{v_{2,t} : 0 \le t \le T\}$, such that

$$
V_t = V_0 + \int_0^t v_{1,s}ds + \int_0^t v_{2,s}dW_s^0 \tag{3.4}
$$

for all $t \ge 0$. The processes of the above form for which $\{v_{1,t} : 0 \le t \le T\}$ and $\{v_{2,t} : 0 \le t \le T\}$ are simple processes, i.e

$$
v_{i,t} = F_i \mathbb{I}_{[t_1,\,t_2]}(t) \tag{3.5}
$$

for all $0 \le t \le T$ and $i \in \{1,\,2\}$, with each $F_i$ being $\mathcal{F}^{W^0}_{t_1}$-measurable, span a linear subspace $\tilde{\mathbb{V}}$ which is dense in $\mathbb{V}$ under the $L^2$ norm. By using the estimate (3.2), for any $p > 0$ and any $T > 0$ we can easily obtain

$$
\int_0^T \left\| h^p\left(\sigma^{1,1}_{\frac{t}{\epsilon}}\right) u^{\epsilon}(t,\,\cdot) \right\|^2_{L^2_{\sigma,\mathcal{C}}(\mathbb{R}^+ \times \Omega)} dt \le TC^{2p}\left\|u_0\right\|^2_{L^2(\mathbb{R}^+)} \tag{3.6}
$$

It follows that for any sequence $\epsilon_n \to 0^+$, there exists a subsequence $\{\epsilon_{k_n} : n \in \mathbb{N}\}$, such that $h^p\left(\sigma^{1,1}_{\frac{\cdot}{\epsilon}}\right)u^{\epsilon}(\cdot,\,\cdot)$ converges weakly to some $u_p(\cdot,\cdot)$ in $L^2_{\sigma,\mathcal{C}}([0,\,T] \times \mathbb{R}^+ \times \Omega)$ for all $p \in \{1,\,2\}$. Testing (3.1) against an arbitrary smooth and compactly supported function $f$ of $x \in \mathbb{R}^+$, using Ito's formula for the product of $\int_{\mathbb{R}^+} u^{\epsilon}(\cdot,\,x)\,f(x)dx$ with a process $V_{\cdot} \in \tilde{\mathbb{V}}$ having the form (3.4) - (3.5), and finally taking expectations, we find that:

$$
\begin{aligned}
&\mathbb{E}_{\sigma,\mathcal{C}}\left[V_t \int_{\mathbb{R}^+} u^{\epsilon}(t,\,x)f(x)dx\right] \\
&\quad = \mathbb{E}_{\sigma,\mathcal{C}}\left[V_0 \int_{\mathbb{R}^+} u_0(x)f(x)dx\right] + r\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[V_s \int_{\mathbb{R}^+} u^{\epsilon}(s,\,x)f'(x)dx\right]ds \\
&\quad\quad - \int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[V_s \int_{\mathbb{R}^+} \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} u^{\epsilon}(s,\,x)f'(x)dx\right]ds
\end{aligned}
$$

$$+\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}\frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2}u^{\epsilon}(s,\,x)f''(x)dx\right]ds$$

$$+\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[v_{1,s}\int_{\mathbb{R}^+}u^{\epsilon}(s,\,x)f(x)dx\right]ds$$

$$+\rho_{1,1}\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)u^{\epsilon}(s,\,x)f'(x)dx\right]ds \tag{3.7}$$

for all $t \leq T$. Thus, setting $\epsilon = \epsilon_{k_n}$ and taking $n \to +\infty$, by the weak convergence results mentioned above we obtain

$$\mathbb{E}_{\sigma,\mathcal{C}}\left[V_t\int_{\mathbb{R}^+}u(t,\,x)f(x)dx\right]$$

$$= \mathbb{E}_{\sigma,\mathcal{C}}\left[V_0\int_{\mathbb{R}^+}u_0(x)f(x)dx\right] - r\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f'(x)dx\right]ds$$

$$+\frac{1}{2}\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u_2(s,\,x)f'(x)dx\right]ds$$

$$+\frac{1}{2}\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u_2(s,\,x)f''(x)dx\right]ds$$

$$+\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[v_{1,s}\int_{\mathbb{R}^+}u(s,\,x)f(x)dx\right]ds$$

$$+\rho_{1,1}\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}u_1(s,\,x)f'(x)dx\right]ds \tag{3.8}$$

for all $0 \leq t \leq T$. The convergence of the terms in the RHS of (3.7) holds pointwise in $t$, while the one term in the LHS converges weakly. Since we can easily find uniform bounds for all the terms in (3.7) (by using (3.6)), the dominated convergence theorem implies that all the weak limits coincide with the corresponding pointwise limits, which gives (3.8) as a limit of (3.7) both weakly and pointwise in $t$. It is clear then that $\mathbb{E}_{\sigma,\mathcal{C}}\left[V_t\int_{\mathbb{R}^+}u(t,\,x)f(x)dx\right]$ is differentiable in $t$ (in a $W^{1,1}$ sense). Next, we can also check that $\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{i,t}\int_{\mathbb{R}^+}u^{\epsilon_{k_n}}(t,\,x)f(x)dx\right]$ converges to $\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{i,t}\int_{\mathbb{R}^+}u(t,\,x)f(x)dx\right]$ for $i \in \{1,\,2\}$, both weakly and pointwise in $t \in [0,\,T]$, while the limits are also differentiable in $t$ everywhere except the two jump points $t_1$ and $t_2$. This follows from the fact that everything is zero outside $[t_1,\,t_2]$, while both $v_{1,\cdot}$ and $v_{2,\cdot}$ are constant in $t$ and thus of the form (3.4) - (3.5) if we restrict on that interval. Observe now that we can write (3.7) as

$$\mathbb{E}_{\sigma,\mathcal{C}}\left[V_t\int_{\mathbb{R}^+}u^{\epsilon}(t,\,x)f(x)dx\right]$$

$$= \mathbb{E}_{\sigma,\mathcal{C}}\left[V_0\int_{\mathbb{R}^+}u_0(x)f(x)dx\right] + r\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u^{\epsilon}(s,\,x)f'(x)dx\right]ds$$

$$-\int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2}$$

$$\times\left(\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u^{\epsilon}(s,\,x)f'(x)dx\right] - \mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f'(x)dx\right]\right)ds$$

$$-\int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2}\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f'(x)dx\right]ds$$

$$
\begin{aligned}
&+\int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2} \\
&\qquad \times\left(\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u^\epsilon(s,\,x)f''(x)dx\right]-\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f''(x)dx\right]\right)ds \\
&+\int_0^t \frac{h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)}{2}\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f''(x)dx\right]ds \\
&+\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[v_{1,s}\int_{\mathbb{R}^+}u^\epsilon(s,\,x)f(x)dx\right]ds \\
&+\rho_{1,1}\int_0^t h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right) \\
&\qquad \times\left(\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}u^\epsilon(s,\,x)f'(x)dx\right]-\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}u(s,\,x)f'(x)dx\right]\right)ds \\
&+\rho_{1,1}\int_0^t h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}u(s,\,x)f'(x)dx\right]ds
\end{aligned}
\tag{3.9}
$$

Then we have

$$
\begin{aligned}
&\left|\int_0^t h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)\left(\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}u^\epsilon(s,\,x)f'(x)dx\right]-\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}u(s,\,x)f'(x)dx\right]\right)ds\right| \\
&\qquad \le C\int_0^t\left|\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}u^\epsilon(s,\,x)f'(x)dx\right]-\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}u(s,\,x)f'(x)dx\right]\right|ds
\end{aligned}
$$

which tends to zero (when $\epsilon=\epsilon_{k_n}$ and $n\to+\infty$) by the dominated convergence theorem, since the quantity inside the last integral converges pointwise to zero as we have mentioned earlier, while it can be dominated by using (3.6). The same argument is used to show that 4th and 6th terms in (3.9) tend also to zero in the same subsequence. Finally, for any term of the form

$$
\int_0^t h^p\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f^{(m)}(x)dx\right]ds
$$

for $p,m\in\{0,\,1,\,2\}$, we can recall the differentiability of the second factor inside the integral (which was mentioned earlier) and then use integration by parts to write it as:

$$
\begin{aligned}
&\int_0^t h^p\left(\sigma^{1,1}_{\frac{w}{\epsilon}}\right)dw\left(\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(t,\,x)f^{(m)}(x)dx\right]\right) \\
&\qquad -\int_0^t\int_0^s h^p\left(\sigma^{1,1}_{\frac{w}{\epsilon}}\right)dw\left(\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f^{(m)}(x)dx\right]\right)'ds
\end{aligned}
$$

which converges, by the positive recurrence property, to the quantity

$$
\begin{aligned}
&t\mathbb{E}\left[h^p\left(\sigma^{1,*}\right)|\mathcal{C}\right]\left(\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(t,\,x)f^{(m)}(x)dx\right]\right) \\
&\qquad -\int_0^t s\mathbb{E}\left[h^p\left(\sigma^{1,*}\right)|\mathcal{C}\right]\left(\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f^{(m)}(x)dx\right]\right)'ds.
\end{aligned}
$$

By using integration by parts once more, the last is equal to

$$\mathbb{E}\left[h^p\left(\sigma^{1,*}\right)|\mathcal{C}\right]\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f^{(m)}(x)dx\right]ds \tag{3.10}$$

The last convergence result holds also if we replace $V_\cdot$ by $v_{1,\cdot}$ or $v_{2,\cdot}$, as we can show by following exactly the same steps in the subinterval $[t_1,\,t_2]$ (where $v_{i,\cdot}$ is supported for $i\in\{1,\,2\}$ and where we have differentiability that allows integration by parts).

If we set now $\epsilon=\epsilon_{k_n}$ in (3.9), take $n\to+\infty$, and substitute all the above convergence results, we obtain

$$\begin{aligned}
&\mathbb{E}_{\sigma,\mathcal{C}}\left[V_t\int_{\mathbb{R}^+}u(t,\,x)f(x)dx\right]\\
&\quad=\mathbb{E}_{\sigma,\mathcal{C}}\left[V_0\int_{\mathbb{R}^+}u_0(x)f(x)dx\right]+\left(r-\frac{\sigma_{1,1}^2}{2}\right)\int_0^t\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f'(x)dx\right]ds\\
&\qquad+\frac{\sigma_{1,1}^2}{2}\int_0^t\mathbb{E}_{\sigma,\mathcal{C}}\left[V_s\int_{\mathbb{R}^+}u(s,\,x)f''(x)dx\right]ds\\
&\qquad+\int_0^t\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{1,s}\int_{\mathbb{R}^+}u(s,\,x)f(x)dx\right]ds\\
&\qquad+\rho_{1,1}\sigma_{2,1}\int_0^t\mathbb{E}_{\sigma,\mathcal{C}}\left[v_{2,s}\int_{\mathbb{R}^+}u(s,\,x)f'(x)dx\right]ds.
\end{aligned}\tag{3.11}$$

Since $\tilde{\mathbb{V}}$ is dense in $\mathbb{V}$, for a fixed $t\le T$, we can have (3.11) for any square-integrable Martingale $\{V_s:0\le s\le t\}$, for which we obviously have $v_{1,s}=0$ for all $0\le s\le t$. Next, we denote by $R_u(t,\,x)$ the RHS of (3.3). Using then Ito's formula for the product of $\int_{\mathbb{R}^+}R_u(t,\,x)f(x)dx$ with $V_t$, subtracting $V_t\int_{\mathbb{R}^+}u(t,\,x)f(x)dx$ from both sides, taking expectations and finally substituting from (3.11), we find that

$$\mathbb{E}_{\sigma,\mathcal{C}}\left[V_t\left(\int_{\mathbb{R}^+}R_u(t,\,x)f(x)dx-\int_{\mathbb{R}^+}u(t,\,x)f(x)dx\right)\right]=0$$

for any $t\le T$. By the Martingale Representation Theorem, for that fixed $t\le T$, $V_t$ can be taken equal to the indicator $\mathbb{I}_{\mathcal{E}_t}$, where we define

$$\mathcal{E}_t=\left\{\omega\in\Omega:\int_{\mathbb{R}^+}R_u(t,\,x)f(x)dx>\int_{\mathbb{R}^+}u(t,\,x)f(x)dx\right\}\tag{3.12}$$

and this allows us to write

$$\mathbb{E}_{\sigma,\mathcal{C}}\left[\mathbb{I}_{\mathcal{E}_t}\left(\int_{\mathbb{R}^+}R_u(t,\,x)f(x)dx-\int_{\mathbb{R}^+}u(t,\,x)f(x)dx\right)\right]=0$$

for all $0\le t\le T$. If we integrate the above for $t\in[0,\,T]$ we obtain that

$$\int_0^T\mathbb{E}_{\sigma,\mathcal{C}}\left[\mathbb{I}_{\mathcal{E}_t}\left(\int_{\mathbb{R}^+}R_u(t,\,x)f(x)dx-\int_{\mathbb{R}^+}u(t,\,x)f(x)dx\right)\right]dt=0$$

where the quantity inside the expectation is always non-negative and becomes zero only when $\mathbb{I}_{\mathcal{E}_t}=0$. This implies that $\int_{\mathbb{R}^+}R_u(t,\,x)f(x)dx\le\int_{\mathbb{R}^+}u(t,\,x)f(x)dx$ almost everywhere, and working in the same way with the indicator of the complement $\mathbb{I}_{\mathcal{E}_t^c}$ we can deduce the opposite inequality as well. Thus, we must have $\int_{\mathbb{R}^+} R_u(t,\, x) f(x) dx = \int_{\mathbb{R}^+} u(t,\, x) f(x) dx$ almost everywhere, and since the function $f$ is an arbitrary smooth function with compact support, we can deduce that $R_u$ coincides with $u$ almost everywhere, which gives (3.3).

If $h$ is bounded from below, we can use (3.2) to obtain a uniform (independent from $\epsilon$) bound for the $H^1_0(\mathbb{R}^+) \times L^2_{\sigma, \mathcal{C}}(\Omega \times [0,\, T])$ norm of $u^{\epsilon_{k_n}}$, which implies that in a further subsequence, the weak convergence to $u$ holds also in that Sobolev space, in which (3.3) has a unique solution (see [6]). This implies convergence of $u^{\epsilon}$ to the unique solution of (3.3) in $H^1_0(\mathbb{R}^+) \times L^2_{\sigma, \mathcal{C}}(\Omega \times [0,\, T])$, as $\epsilon \to 0^+$ □

# 4 Fast mean-reversion - large vol-of-vol: A different approach

As we have already mentioned, our final goal is to find good approximations for mass probabilities of the form (1.3), which means that first we need to prove a convergence result of the form

$$\mathbb{P}\left(\mathbb{P}\left(X^{1,\epsilon}_t > 0 \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right) \in (a,\, b)\right) \to \mathbb{P}\left(\mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right) \in (a,\, b)\right) \quad (4.1)$$

as $\epsilon \to 0^+$, where $X^{1,\epsilon}_\cdot$ stands for the first asset price process with volatility path $\sigma^{1,1}_{\frac{\cdot}{\epsilon}}$ and coefficient vector $C'_1$ (as defined in the previous section), while $X^{1,*}_\cdot$ stands for some other stochastic process. This is by definition the convergence in distribution of the random mass over $\mathbb{R}^+$, i.e $\mathbb{P}\left(X^{1,\epsilon}_t > 0 \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right)$, to $\mathbb{P}\left(X^{1,*}_t > 0 \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right)$. In this section we are going to prove the more general

$$\mathbb{P}\left(\mathbb{P}\left(X^{1,\epsilon}_t \in \mathcal{I} \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right) \in (a,\, b)\right) \to \mathbb{P}\left(\mathbb{P}\left(X^{1,*}_t \in \mathcal{I} \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right) \in (a,\, b)\right) \quad (4.2)$$

for some process $X^{1,*}_\cdot$ and any interval $\mathcal{I} = (0, U]$ with $U \in (0, +\infty]$. However, what is $X^{1,*}_\cdot$ going to be? Obviously, if $X^{1,*}_\cdot$ coincides (in distribution) with the process $X^{1,w}_\cdot$ whose density given $\left(W^0_\cdot,\, \sigma^{1,1}_\cdot,\, C'_1,\, \mathcal{G}\right)$ (on the set of positive real numbers) is the weak limit $u$ of $u^\epsilon$ derived in the previous section, we have a very stable result that allows us to hope for better approximations. Otherwise, we cannot expect any convergence of the random mass over $\mathcal{I}$, better than convergence in distribution. Indeed, a limit in probability has to coincide with the limit in distribution, and in this case it has to be also a strong $L^2$ limit in some sequence $\epsilon_n \downarrow 0$ (because it will be a $\mathbb{P}$ - almost surely limit in some sequence, and then we can apply the Dominated Convergence Theorem), so it has to be a weak $L^2$ limit as well. Then, for $\Xi = \mathbb{P}\left(X^{1,w}_t \in \mathcal{I} \,|\, W^0_\cdot,\, \sigma^{1,1}_\cdot,\, \mathcal{G}\right) - \mathbb{P}\left(X^{1,*}_t \in \mathcal{I} \,|\, W^0_\cdot,\, \sigma^{1,1}_\cdot,\, \mathcal{G}\right)$, assuming that the coefficient vector is the same constant vector for all the assets we have

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \mathbb{E}\left[\Xi\left(\mathbb{P}\left(X^{1,\epsilon_n}_t \in \mathcal{I} \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right) - \mathbb{P}\left(X^{1,*}_t \in \mathcal{I} \,|\, W^0_\cdot,\, B^0_\cdot,\, \mathcal{G}\right)\right)\right] \\
&= \lim_{n\to+\infty} \mathbb{E}\left[\Xi\left(\mathbb{P}\left(X^{1,\epsilon_n}_t \in \mathcal{I} \,|\, W^0_\cdot,\, \sigma^{1,1}_\cdot,\, \mathcal{G}\right) - \mathbb{P}\left(X^{1,*}_t \in \mathcal{I} \,|\, W^0_\cdot,\, \sigma^{1,1}_\cdot,\, \mathcal{G}\right)\right)\right] \\
&= \lim_{n\to+\infty} \mathbb{E}\left[\int_0^{+\infty} \Xi \mathbb{I}_{\mathcal{I}}(x) u^{\epsilon_n}(t, x) dx - \Xi \mathbb{P}\left(X^{1,*}_t \in \mathcal{I} \,|\, W^0_\cdot,\, \sigma^{1,1}_\cdot,\, \mathcal{G}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^{+\infty} \Xi \mathbb{I}_{\mathcal{I}}(x) u(t,x)dx - \Xi \mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}\right)\right] \\
&= \mathbb{E}\left[\Xi\left(\mathbb{P}\left(X_t^{1,w} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}\right) - \mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}\right)\right)\right] \\
&= \mathbb{E}\left[\left(\mathbb{P}\left(X_t^{1,w} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}\right) - \mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, \sigma_\cdot^{1,1},\, \mathcal{G}\right)\right)^2\right] \\
&> 0
\end{aligned}
$$

for any bounded interval $\mathcal{I}$, which is a contradiction.

When $X_\cdot^{1,*}$ and $X_\cdot^{1,w}$ do not coincide and we only have weak convrgence, good approximations of the RHS of (4.2) can only be obtained by studying the asymptotic behaviour of the distribution of the random mass over $\mathcal{I}$, and not the asymptotic behaviour of that mass itself, which makes our problem really challenging since we have no idea how this distribution looks like. Unfortunately, we will see that $X_\cdot^{1,*}$ is only equal to $X_\cdot^{1,w}$ when we have no market noise in our model, which means that we can only hope for nice convergence results when the most important feature of the model we are studying is removed! Moreover, the convergence in distribution result of this section can only be obtained when the volatility coefficients are deterministic and the same for each asset, i.e $(\kappa_i,\, \theta_i,\, v_i,\, \rho_{2,i}) = (\kappa,\, \theta,\, v,\, \rho_2)$ for all $i \in \mathbb{N}$. In the general case of i.i.d vectors $(\kappa_i,\, \theta_i,\, v_i,\, \rho_{2,i})$ for $i \in \mathbb{N}$, we will see that we cannot even hope for the existence of a process $X_\cdot^{1,*}$ satisfying (4.2). The main result of this section is given in the following theorem

**Theorem 4.1.** *Suppose that $(\kappa_i,\, \theta_i,\, v_i,\, \rho_{2,i}) = (\kappa,\, \theta,\, v,\, \rho_2)$ for all $i \in \mathbb{N}$, which is a deterministic 4-dimensional vector, and consider the stochastic process $Y_\cdot^{1,*}$ which is given by $Y_t^{1,*} = X_0^1 + \left(r_1 - \frac{\sigma_{1,1}^2}{2}\right)t + \tilde{\rho}_{1,1}\sigma_{1,1}W_t^0 + \sqrt{1-\tilde{\rho}_{1,1}^2}\sigma_{1,1}W_t^1$ for all $t \geq 0$, with $\tilde{\rho}_{1,1} = \rho_{1,1}\frac{\tilde{\sigma}}{\sigma_{1,1}}$ for some $\tilde{\sigma} \in [\sigma_{2,1},\, \sigma_{1,1}]$, where $\sigma_{2,1},\, \sigma_{1,1}$ are defined in Theorem 3.1. Define then $X_t^{1,*} = Y_{t\wedge\tau_{1,*}}^{1,*}$, for $\tau_{1,*} = \inf\{s \geq 0 : Y_s^{1,*} \leq 0\}$. Then, if the function $h$ is bounded and the function $g$ has the positive recurrence property, we have that $\mathbb{P}\left(X_t^{1,\epsilon} \in (0,U] \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)$ converges in distribution to $\mathbb{P}\left(X_t^{1,*} \in (0,U] \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)$ as $\epsilon \to 0^+$, for any $U \in (0,+\infty]$.*

*Proof.* To have the desired convergence in distribution, we need to show that for every bounded and continuous function $G : \mathbb{R} \to \mathbb{R}$ we have:

$$
\mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)\right)\right] \to \mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)\right)\right] \tag{4.3}
$$

as $\epsilon \to 0^+$, where $\mathcal{I} = (0,+\infty]$. Observe now that since the conditional probabilities describing the default mass take values in the compact interval $[0,1]$, it is equivalent to have the above for all continuous $G : [0,1] \to \mathbb{R}$. We actually need to have this only when $G$ is a polynomial, since in that case, for an arbitrary continuous function $G$ and for a polynomial $P$ such that $|P(x) - G(x)| < \eta$ for all $x \in [0,1]$, we have:

$$
\begin{aligned}
&\left|\mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)\right)\right] - \mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,*} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)\right)\right]\right| \\
&\quad \leq \left|\mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)\right)\right] - \mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,\epsilon} \in \mathcal{I} \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right)\right)\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,\epsilon}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right)\right]-\mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,*}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right)\right]\right|\\
&+\left|\mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,*}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right)\right]-\mathbb{E}\left[G\left(\mathbb{P}\left(X_t^{1,*}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right)\right]\right|\\
\leq\ & 2\eta+\left|\mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,\epsilon}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right)\right]-\mathbb{E}\left[P\left(\mathbb{P}\left(X_t^{1,*}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right)\right]\right|
\end{aligned}
$$

where $\eta$ can be taken as small as we want, and for that fixed $\eta$ the last difference tends to 0 as $\epsilon\to 0^+$. Finally, by linearity we only need to have (4.3) when $G(x)=x^m$ for all $x\in[0,1]$, for some $m\in\mathbb{N}$.

For a given $m\in\mathbb{N}$ now, let $\{X_\cdot^{i,\epsilon}\,:\,1<i\leq m\}$ be a collection of $m-1$ processes with the same distribution as $X_\cdot^{1,\epsilon}$, which are driven by the 4-dimensional Brownian Motions $\left(W_\cdot^0,\,W_\cdot^2,\,B_\cdot^0,\,B_\cdot^2\right)$, $\left(W_\cdot^0,\,W_\cdot^3,\,B_\cdot^0,\,B_\cdot^3\right)$, ..., $\left(W_\cdot^0,\,W_\cdot^m,\,B_\cdot^0,\,B_\cdot^m\right)$. That is, for $1\leq i\leq m$ we define:

$$
Y_t^{i,\epsilon}=Y_0^i+\int_0^t\left(r_i-\frac{h^2\left(\sigma_{\frac{s}{\epsilon}}^{i,1}\right)}{2}\right)ds+\int_0^t h(\sigma_{\frac{s}{\epsilon}}^{i,1})\left(\sqrt{1-\rho_{1,i}^2}dW_s^i+\rho_{1,i}dW_s^0\right)
$$

with

$$
\sigma_t^{i,1}=\sigma_0^{i,1}+\kappa\int_0^t\left(\theta-\sigma_s^{i,1}\right)ds+v\int_0^t g\left(\sigma_s^{i,1}\right)\rho_2 dB_s^0+v\int_0^t g\left(\sigma_s^{i,1}\right)\sqrt{1-\rho_2^2}dB_s^i
$$

for all $t\geq 0$, and then we define $X_t^{i,\epsilon}=Y_{t\wedge\tau^{i,\epsilon}}^{i,\epsilon}$ for $\tau^{i,\epsilon}=\inf\{s\geq 0\,:\,Y_s^{i,\epsilon}\leq 0\}$. The $m$ processes $\{X_\cdot^{i,\epsilon}\,:\,1\leq i\leq m\}$ are obviously pairwise i.i.d when the information contained in $W_\cdot^0, B_\cdot^0$ and $\mathcal{G}$ is given. Therefore we can write:

$$
\begin{aligned}
&\mathbb{E}\left[g\left(\mathbb{P}\left(X_t^{1,\epsilon}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right)\right]\\
&\quad=\mathbb{E}\left[\mathbb{P}^m\left(X_t^{1,\epsilon}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right]\\
&\quad=\mathbb{E}\left[\mathbb{P}\left(X_t^{1,\epsilon}\in\mathcal{I},\,X_t^{2,\epsilon}\in\mathcal{I},\,...,\,X_t^{m,\epsilon}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right]\\
&\quad=\mathbb{P}\left(X_t^{1,\epsilon}\in\mathcal{I},\,X_t^{2,\epsilon}\in\mathcal{I},\,...,\,X_t^{m,\epsilon}\in\mathcal{I}\right)\\
&\quad=\mathbb{P}\left(\left(\min_{1\leq i\leq m}\min_{0\leq s\leq t}Y_s^{i,\epsilon},\,\max_{1\leq i\leq m}Y_t^{i,\epsilon}\right)\in(0,\,+\infty)\times(-\infty,U]\right)\qquad(4.4)
\end{aligned}
$$

Next, we consider the collection of processes $\{X_\cdot^{i,*}\,:\,1\leq i\leq m\}$, which are defined for $i>1$ exactly as for $i=1$, i.e $X_t^{i,*}=Y_{t\wedge\tau_{i,*}}^{i,*}$ for $\tau_{i,*}=\inf\{s\geq 0\,:\,Y_s^{i,*}\leq 0\}$, where each $Y_\cdot^{i,*}$ is defined as

$$
Y_t^{i,*}=X_0^i+\left(r_i-\frac{\sigma_{1,1}^2}{2}\right)t+\tilde{\rho}_{1,i}\sigma_{1,1}W_t^0+\sqrt{1-\tilde{\rho}_{1,i}^2}\sigma_{1,1}W_t^i
$$

for all $t\geq 0$, with $\tilde{\rho}_{1,i}=\rho_{1,i}\frac{\tilde{\sigma}}{\sigma_{1,1}}$ for some $\tilde{\sigma}\in[\sigma_{2,1},\,\sigma_{1,1}]$ (which will be chosen later) and all $i\in\{2,\,...,\,m\}$. Again, it is easy to check that the above processes are pairwise i.i.d processes when the information contained in $W_\cdot^0, B_\cdot^0$ and $\mathcal{G}$ is given. Thus, we can write

$$
\mathbb{E}\left[g\left(\mathbb{P}\left(X_t^{1,*}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right)\right]
$$

$$= \mathbb{E}\left[\mathbb{P}^m\left(X_t^{1,*}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right]$$
$$= \mathbb{E}\left[\mathbb{P}\left(X_t^{1,*}\in\mathcal{I},\,X_t^{2,*}\in\mathcal{I},\,...,\,X_t^{m,*}\in\mathcal{I}\,|\,W_\cdot^0,\,B_\cdot^0,\,\mathcal{G}\right)\right]$$
$$= \mathbb{P}\left(X_t^{1,*}\in\mathcal{I},\,X_t^{2,*}\in\mathcal{I},\,...,\,X_t^{m,*}\in\mathcal{I}\right)$$
$$= \mathbb{P}\left(\left(\min_{1\le i\le m}\min_{0\le s\le t}Y_s^{i,*},\,\max_{1\le i\le m}Y_t^{i,*}\right)\in(0,\,+\infty)\times(-\infty,U]\right). \quad (4.5)$$

Then, (4.4) and (4.5) show that the result we want to prove has been reduced to the convergence of $\left(\min_{1\le i\le m}\min_{0\le s\le t}Y_s^{i,\epsilon},\,\max_{1\le i\le m}Y_t^{i,\epsilon}\right)$ to $\left(\min_{1\le i\le m}\min_{0\le s\le t}Y_s^{i,*},\,\max_{1\le i\le m}Y_t^{i,*}\right)$, in distribution as $\epsilon\to 0^+$ (since the probability that any of the $m$ minimums equals zero is zero, as the minimum of any Gaussian process is always continuously distributed, while $Y_\cdot^{i,\epsilon}$ is obviously Gaussian for any given path of $\sigma_\cdot^{i,1}$).

Let now $C\left([0,\,t]\,;\mathbb{R}^m\right)$ be the classical Wiener space of continuous functions defined on $[0,\,t]$ and taking values in $\mathbb{R}^k$ (i.e the space of these functions equipped with the supremum norm and the Wiener probability measure), and observe that $\min_{1\le i\le m}p_i\left(\min_{0\le s\le t}\cdot(s)\right)$ defined on $C\left([0,\,t]\,;\mathbb{R}^m\right)$, where $p_i$ stands for the projection on the $i$-th axis, is a continuous functional. Indeed, for any two continuous functions $f_1,\,f_2$ in $C\left([0,\,t]\,;\mathbb{R}^m\right)$, we have:

$$\left|\min_{1\le i\le m}p_i\left(\min_{0\le s\le t}f_1(s)\right)-\min_{1\le i\le m}p_i\left(\min_{0\le s\le t}f_2(s)\right)\right| = \left|p_{i_1}\left(f_1(s_1)\right)-p_{i_2}\left(f_2(s_2)\right)\right|$$

for some $s_1,s_2\in[0,\,t]$ and $1\le i_1,i_2\le m$, and without loss of generality we may assume that the difference inside the last absolute value is nonnegative. Moreover we have:

$$p_{i_1}\left(f_1(s_1)\right)=\min_{1\le i\le m}p_i\left(\min_{0\le s\le t}f_1(s)\right)\le p_{i_2}\left(f_1(s_2)\right)$$

and thus we have:

$$\begin{aligned}\left|\min_{1\le i\le m}p_i\left(\min_{0\le s\le t}f_1(s)\right)-\min_{1\le i\le m}p_i\left(\min_{0\le s\le t}f_2(s)\right)\right| &= p_{i_1}\left(f_1(s_1)\right)-p_{i_2}\left(f_2(s_2)\right)\\ &\le p_{i_2}\left(f_1(s_2)\right)-p_{i_2}\left(f_2(s_2)\right)\\ &\le \left|p_{i_2}\left(f_1(s_2)\right)-p_{i_2}\left(f_2(s_2)\right)\right|\\ &\le \|f_1-f_2\|_{C([0,\,t];\mathbb{R}^m)}\end{aligned}$$

Obviously, $\max_{1\le i\le m}p_i\left(\cdot\,(t)\right)$ defined on $C\left([0,\,t]\,;\mathbb{R}^m\right)$ is also continuous (as the maximum of finitely many evaluation functionals). Therefore, our problem is finally reduced to showing that $\left(Y_\cdot^{1,\epsilon},\,Y_\cdot^{2,\epsilon},\,...,\,Y_\cdot^{m,\epsilon}\right)$ converges in distribution to $\left(Y_\cdot^{1,*},\,Y_\cdot^{2,*},\,...,\,Y_\cdot^{m,*}\right)$ in the space $C\left([0,\,t]\,;\mathbb{R}^m\right)$, as $\epsilon\to 0^+$.

We will now follow the standard way for showing convergence in distribution results like the above: We will show first that a limit in distribution exists as as $\epsilon\to 0^+$ by using a tightness argument, and then we will try to characterize the limits of the finite dimensional distributions. To show tightness of the laws of $\left(Y_\cdot^{1,\epsilon},\,Y_\cdot^{2,\epsilon},\,...,\,Y_\cdot^{m,\epsilon}\right)$ for $\epsilon\in\mathbb{R}^+$, which implies the desired convergence in distribution, we recall a special case of

Theorem 7.2 in [7] for continuous processes, according to which it suffices to prove that for a given $\eta > 0$, there exist some $\delta > 0$ and $N > 0$ such that:

$$\mathbb{P}\left(\left\|\left(Y_0^{1,\epsilon},\, Y_0^{2,\epsilon},\, ...,\, Y_0^{m,\epsilon}\right)\right\|_{\mathbb{R}^m} > N\right) \leq \eta \tag{4.6}$$

and

$$\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\, |s_1-s_2|\leq\delta} \left\|\left(Y_{s_1}^{1,\epsilon},\, Y_{s_1}^{2,\epsilon},\, ...,\, Y_{s_1}^{m,\epsilon}\right) - \left(Y_{s_2}^{1,\epsilon},\, Y_{s_2}^{2,\epsilon},\, ...,\, Y_{s_2}^{m,\epsilon}\right)\right\|_{\mathbb{R}^m} > \eta\right) \leq \eta \tag{4.7}$$

for all $\epsilon > 0$. (4.6) can easily be achieved for some $N > 0$ since $\left(Y_0^{1,\epsilon},\, Y_0^{2,\epsilon},\, ...,\, Y_0^{m,\epsilon}\right) = \left(X_0^1,\, X_0^2,\, ...,\, X_0^m\right)$, which is independent of $\epsilon$ and almost surely finite (the sum of the probabilities that the norm of this vector belongs to $[n,\, n+1]$ over $n \in \mathbb{N}$ is a convergent series and thus, by Cauchy criteria, the same sum but for $n \geq N$ tends to zero as $N$ tends to infinity). For (4.7) now, observe that $\|\cdot\|_{\mathbb{R}^m}$ can be any of the standard equivalent $L^p$ norms of $\mathbb{R}^m$, and we choose it to be $L^\infty$. Then we have:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\, |s_1-s_2|\leq\delta} \left\|\left(Y_{s_1}^{1,\epsilon},\, Y_{s_1}^{2,\epsilon},\, ...,\, Y_{s_1}^{m,\epsilon}\right) - \left(Y_{s_2}^{1,\epsilon},\, Y_{s_2}^{2,\epsilon},\, ...,\, Y_{s_2}^{m,\epsilon}\right)\right\|_{\mathbb{R}^m} > \eta\right) \\
&\qquad = \mathbb{P}\left(\cup_{i=1}^m \{\sup_{0\leq s_1,s_2\leq t,\, |s_1-s_2|\leq\delta} \left|Y_{s_1}^{i,\epsilon} - Y_{s_2}^{i,\epsilon}\right| > \eta\}\right) \\
&\qquad \leq \sum_{i=1}^m \mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\, |s_1-s_2|\leq\delta} \left|Y_{s_1}^{i,\epsilon} - Y_{s_2}^{i,\epsilon}\right| > \eta\right) \\
&\qquad = m\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\, |s_1-s_2|\leq\delta} \left|Y_{s_1}^{1,\epsilon} - Y_{s_2}^{1,\epsilon}\right| > \eta\right)
\end{aligned} \tag{4.8}$$

and since it is well known that the Ito integral $\int_0^t h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)\left(\sqrt{1-\rho_1^2}dW_s^1 + \rho_1 dW_s^0\right)$ can be written as $\tilde{W}_{\int_0^t h^2\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)ds}$, where $\tilde{W}_{\cdot}$ is another standard Brownian motion, by writing $\Delta\tilde{W}^h\left(s_1, s_2\right)$ for the difference $\tilde{W}_{\int_0^{s_1} h^2\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)ds} - \tilde{W}_{\int_0^{s_2} h^2\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)}$ for all $s_1, s_2 > 0$ and by denoting the maximum of $h$ by $M$, we also have:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\, |s_1-s_2|\leq\delta} \left|Y_{s_1}^{1,\epsilon} - Y_{s_2}^{1,\epsilon}\right| > \eta\right) \\
&\quad = \mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\, |s_1-s_2|\leq\delta} \left|\int_{s_2}^{s_1}\left(r - \frac{h^2\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)}{2}\right)ds + \left(\Delta\tilde{W}^h\left(s_1, s_2\right)\right)\right| > \eta\right) \\
&\quad \leq \mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\, |s_1-s_2|\leq\delta} \left|\int_{s_2}^{s_1}\left(r - \frac{h^2\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)}{2}\right)ds\right| > \frac{\eta}{2}\right) \\
&\qquad + \mathbb{P}\left(\sup_{0\leq s_1,s_2\leq t,\, |s_1-s_2|\leq\delta} \left|\left(\tilde{W}_{\int_0^{s_1} h^2\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)ds} - \tilde{W}_{\int_0^{s_2} h^2\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)ds}\right)\right| > \frac{\eta}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{P}\left(\delta\,(r+M)>\frac{\eta}{2}\right)\\
&\qquad +\mathbb{P}\left(\sup_{0\leq s_3,s_4\leq M^2t,\,|s_3-s_4|\leq M^2\delta}\left|\left(\tilde{W}_{s_3}-\tilde{W}_{s_4}\right)\right|>\frac{\eta}{2}\right)
\end{aligned}
\tag{4.9}
$$

since $\left|\int_a^b h^2\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)ds\right|\leq M^2\,|a-b|$ for all $a,b\in\mathbb{R}^+$. The first of the last two probabilities is clearly zero for $\delta<\frac{\eta}{2(r+M)}$, while the second one can also be made arbitrarily small for small enough $\delta$, since by a well known result about the modulus of continuity of a Brownian motion (see [26]) the supremum within that probability converges almost surely (and thus also in probability) to 0 as fast as $M\sqrt{2\delta\log\left(\frac{1}{M^2\delta}\right)}$. Plugging these in (4.8) we deduce that (4.7) is also satisfied and we have the desired tightness result, which implies that $\left(Y^{1,\epsilon}_{\cdot},\,Y^{2,\epsilon}_{\cdot},\,...,\,Y^{m,\epsilon}_{\cdot}\right)$ converges in distribution to some limit $\left(Y^{1,0}_{\cdot},\,Y^{2,0}_{\cdot},\,...,\,Y^{m,0}_{\cdot}\right)$.

To conclude our proof, we need to show that $\left(Y^{1,0}_{\cdot},\,Y^{2,0}_{\cdot},\,...,\,Y^{m,0}_{\cdot}\right)$ coincides with $\left(Y^{1,*}_{\cdot},\,Y^{2,*}_{\cdot},\,...,\,Y^{m,*}_{\cdot}\right)$. Since both $m$-dimensional processes are uniquely determined by their finite-dimensional distributions, and since evaluation functionals on $C\left([0,\,t]\,;\mathbb{R}^m\right)$ preserve convergences in distribution (as continuous functionals), we only need to show that for any fixed $(i_1,\,i_2,\,...,\,i_\ell)\in\{1,\,2,\,...,\,m\}^\ell$, any fixed $(t_1,\,t_2,\,...,\,t_\ell)\in(0,\,+\infty)^\ell$, and any fixed continuous and bounded function $q:\mathbb{R}^\ell\to\mathbb{R}$, for an arbitrary $\ell\in\mathbb{N}$, we have

$$
\mathbb{E}\left[q\left(Y^{i_1,\epsilon}_{t_1},\,Y^{i_2,\epsilon}_{t_2},\,...,\,Y^{i_\ell,\epsilon}_{t_\ell}\right)\right]\to\mathbb{E}\left[q\left(Y^{i_1,*}_{t_1},\,Y^{i_2,*}_{t_2},\,...,\,Y^{i_\ell,*}_{t_\ell}\right)\right]
$$

as $\epsilon\to0^+$. Due to the Dominated Convergence Theorem, the above follows if we are able to show that

$$
\begin{aligned}
&\lim_{\epsilon\to0^+}\mathbb{E}\left[q\left(Y^{i_1,\epsilon}_{t_1},\,Y^{i_2,\epsilon}_{t_2},\,...,\,Y^{i_\ell,\epsilon}_{t_\ell}\right)|\sigma^{i_1,1}_{\cdot},\,\sigma^{i_2,1}_{\cdot},\,...,\,\sigma^{i_\ell,1}_{\cdot},\,\mathcal{C}\right]\\
&\qquad=\lim_{\epsilon\to0^+}\mathbb{E}\left[q\left(Y^{i_1,*}_{t_1},\,Y^{i_2,*}_{t_2},\,...,\,Y^{i_\ell,*}_{t_\ell}\right)|\sigma^{i_1,1}_{\cdot},\,\sigma^{i_2,1}_{\cdot},\,...,\,\sigma^{i_\ell,1}_{\cdot},\,\mathcal{C}\right]
\end{aligned}
$$

$\mathbb{P}$- almost surely. However, when the information contained in $\sigma^{i_1,1}_{\cdot},\,\sigma^{i_2,1}_{\cdot},\,...,\,\sigma^{i_\ell,1}_{\cdot}$ and $\mathcal{C}$ is given, both $\left(Y^{i_1,\epsilon}_{t_1},\,Y^{i_2,\epsilon}_{t_2},\,...,\,Y^{i_\ell,\epsilon}_{t_\ell}\right)$ and $\left(Y^{i_1,*}_{t_1},\,Y^{i_2,*}_{t_2},\,...,\,Y^{i_\ell,*}_{t_\ell}\right)$ have an $\ell$-dimensional normal distribution. This means that given $\left(\sigma^{i_1,1}_{\cdot},\,\sigma^{i_2,1}_{\cdot},\,...,\,\sigma^{i_\ell,1}_{\cdot}\right)$ and $\mathcal{C}$, we only need to show that as $\epsilon\to0^+$, the mean vector and the covariance matrix of $\left(Y^{i_1,\epsilon}_{t_1},\,Y^{i_2,\epsilon}_{t_2},\,...,\,Y^{i_\ell,\epsilon}_{t_\ell}\right)$ converge to the mean vector and the covariance matrix of $\left(Y^{i_1,*}_{t_1},\,Y^{i_2,*}_{t_2},\,...,\,Y^{i_\ell,*}_{t_\ell}\right)$ respectively. Given $\left(\sigma^{i_1,1}_{\cdot},\,\sigma^{i_2,1}_{\cdot},\,...,\,\sigma^{i_\ell,1}_{\cdot}\right)$, the information contained in $\mathcal{C}$, and a $k\in\{1,\,2,\,...,\,\ell\}$, the $k$-th coordinate of the mean vector of $\left(Y^{i_1,\epsilon}_{t_1},\,Y^{i_2,\epsilon}_{t_2},\,...,\,Y^{i_\ell,\epsilon}_{t_\ell}\right)$ is equal to $X^{i_k}_0+\int_0^{t_k}\left(r_{i_k}-\frac{h^2\left(\sigma^{i_k,1}_{\frac{s}{\epsilon}}\right)}{2}\right)ds$, and by the positive recurrence property it converges as $\epsilon\to0^+$ to $X^{i_k}_0+\left(r_{i_k}-\frac{\sigma^2_{1,1}}{2}\right)t_k$ (since the volatility processes have all the same coefficients and thus the same stationary distributions), which is the $k$-th coordinate of the mean vector of $\left(Y^{i_1,*}_{t_1},\,Y^{i_2,*}_{t_2},\,...,\,Y^{i_\ell,*}_{t_\ell}\right)$. Now we only need to obtain the corresponding convergence result for the covariance matrices of our processes. For some $1 \leq p, q \leq \ell$, given $\left(\sigma_{\cdot}^{i_1,1}, \sigma_{\cdot}^{i_2,1}, ..., \sigma_{\cdot}^{i_\ell,1}\right)$ and the information contained in $\mathcal{C}$, the covariance of $Y_{t_p}^{i_p,\epsilon}$ and $Y_{t_q}^{i_q,\epsilon}$ is equal to

$$\left(\rho_{1,i_p}\rho_{1,i_q} + \delta_{i_p,i_q}\sqrt{1-\rho_{1,i_p}}\sqrt{1-\rho_{1,i_q}}\right)\int_0^{t_p\wedge t_q} h\left(\sigma_{\frac{s}{\epsilon}}^{i_p,1}\right) h\left(\sigma_{\frac{s}{\epsilon}}^{i_q,1}\right) ds$$

while the covariance of $Y_{t_p}^{i_p,*}$ and $Y_{t_q}^{i_q,*}$ is equal to

$$\left(\tilde{\rho}_{1,i_p}\tilde{\rho}_{1,i_q} + \delta_{i_p,i_q}\sqrt{1-\tilde{\rho}_{1,i_p}^2}\sqrt{1-\tilde{\rho}_{1,i_q}^2}\right)\sigma_{1,1}^2 t_p \wedge t_q.$$

This means that for $i_p = i_q = i \in \{1, 2, ..., m\}$ we need to show that

$$\int_0^{t_p\wedge t_q} h^2\left(\sigma_{\frac{s}{\epsilon}}^{i,1}\right) ds \to \sigma_{1,1}^2 t_p \wedge t_q$$

as $\epsilon \to 0^+$, while for $i_p \neq i_q$ we need to show that:

$$\rho_{1,i_p}\rho_{1,i_q}\int_0^{t_p\wedge t_q} h\left(\sigma_{\frac{s}{\epsilon}}^{i_p,1}\right) h\left(\sigma_{\frac{s}{\epsilon}}^{i_q,1}\right) ds \to \tilde{\rho}_{1,i_p}\tilde{\rho}_{1,i_q}\sigma_{1,1}^2 t_p \wedge t_q$$

as $\epsilon \to 0^+$, where $\tilde{\rho}_{1,i}\sigma_{1,1} = \rho_{1,i}\tilde{\sigma}$ for all $i \leq m$. Both convergence results follow from the positive recurrence property for $\tilde{\sigma} = \sqrt{\mathbb{E}\left[h\left(\sigma^{i_p,i_q,1,*}\right) h\left(\sigma^{i_p,i_q,2,*}\right)\right]}$, which does not depend on $i_p$ and $i_q$ since the volatility processes have all the same coefficients and thus the same joint stationary distributions.

It remains to show that $\tilde{\sigma} \in [\sigma_{2,1}, \sigma_{1,1}]$. The upper bound can be obtained by a simple Cauchy-Schwartz inequality, i.e

$$\begin{aligned}\tilde{\sigma} &= \sqrt{\mathbb{E}\left[h\left(\sigma^{1,2,1,*}\right) h\left(\sigma^{1,2,2,*}\right)\right]}\\ &\leq \sqrt{\sqrt{\mathbb{E}\left[h^2\left(\sigma^{1,2,1,*}\right)\right]}\sqrt{\mathbb{E}\left[h^2\left(\sigma^{1,2,2,*}\right)\right]}}\\ &= \sqrt{\sigma_{1,1}\times\sigma_{1,1}}\\ &= \sigma_{1,1} \end{aligned} \tag{4.10}$$

For the lower bound, considering our volatility processes for $i = 1$ and $i = 2$ started from their 1-dimensional stationary distributions independently, we have for any $t, \epsilon \geq 0$

$$\begin{aligned}&\mathbb{E}\left[\frac{1}{t}\int_0^t h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right) h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right) ds\right]\\ &= \frac{1}{t}\int_0^t \mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right) h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right)\right] ds\\ &= \frac{1}{t}\int_0^t \mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)\right]\mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right)\right] ds\\ &\quad + \frac{1}{t}\int_0^t \mathbb{E}\left[\left(h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right) - \mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{1,1}\right)\right]\right)\left(h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right) - \mathbb{E}\left[h\left(\sigma_{\frac{s}{\epsilon}}^{2,1}\right)\right]\right)\right] ds\\ &= \sigma_{2,1}^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)-\mathbb{E}\left[h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)\right]\right)\left(h\left(\sigma^{2,1}_{\frac{s}{\epsilon}}\right)-\mathbb{E}\left[h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)\right]\right)\mid B^0_{\cdot}\right]\right] ds\\
&=\sigma_{2,1}^2+\frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)-\sigma_{2,1}\right)\mid B^0_{\cdot}\right]^2\right] ds\\
&\geq \sigma_{2,1}^2
\end{aligned}
\tag{4.11}
$$

since $\sigma^{1,1}_{\cdot}$ and $\sigma^{2,1}_{\cdot}$ are identically distributed, and also i.i.d when $B^0_{\cdot}$ is given. Taking $\epsilon \to 0^+$ on (4.11) and recalling the positive recurrence property, the definition of $\tilde{\sigma}$, and the Dominated Convergence Theorem on the LHS (since the quantity inside the expectation there is bounded by the square of an upper bound of $h$), we obtain the desired. The proof of the Theorem is now complete. □

*Remark* 4.2. A few comments need to be made about the bounds we have derived for $\tilde{\sigma}$:

1. The upper bound $\tilde{\sigma} \leq \sigma_{1,1}$ is needed to ensure that $\tilde{\rho}_{1,1} \leq 1$, so $X^{1,*}_{\cdot}$ is a real-valued stochastic process. The above proof shows that this bound is only attainable when $\sigma^{i,j,1,*} = \sigma^{i,j,2,*}$ for all $i$ and $j$ with $i \neq j$, which happens only when all the assets share a common stochastic volatility (i.e $\rho_2 = 1$).

2. The lower bound $\tilde{\sigma} \geq \sigma_{2,1}$ can also be shown to be unattainable in general, which means that the conditional (given $\left(W^0_{\cdot},\, B^0_{\cdot},\, \mathcal{G}\right)$) density of $X^{1,*}_t$ on $(0, +\infty)$ does not coincide with the weak limit derived in the previous section, and as we mentioned earlier, this shows that convergence of $\mathbb{P}\left(X^{1,\epsilon}_t > 0 \mid W^0_{\cdot},\, B^0_{\cdot},\, \mathcal{G}\right)$ as $\epsilon \to 0^+$ better than in distribution cannot be expected. Indeed, if we choose $h$ such that its composition $\tilde{h}$ with the square function is strictly increasing and convex, and if $g$ is chosen to be a square root function (thus we are in the CIR volatility case, which is the most common), for any $\alpha > 0$ we have

$$
\begin{aligned}
&\frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)-\sigma_{2,1}\right)\mid B^0_{\cdot}\right]^2\right] ds\\
&\quad=\mathbb{E}\left[\frac{1}{t}\int_0^t \left(\mathbb{E}\left[\tilde{h}\left(\sqrt{\sigma^{1,1}_{\frac{s}{\epsilon}}}\right)\mid B^0_{\cdot}\right]-\sigma_{2,1}\right)^2 ds\right]\\
&\quad\geq \alpha^2\mathbb{E}\left[\frac{1}{t}\int_0^t \mathbb{I}_{\sigma^{B^0,h}_{\frac{s}{\epsilon}}\geq \alpha+\sigma_{2,1}} ds\right]
\end{aligned}
\tag{4.12}
$$

where $\sigma^{B^0,h}_s := \mathbb{E}\left[\tilde{h}\left(\sqrt{\sigma^{1,1}_s}\right)\mid B^0_{\cdot}\right] \geq \tilde{h}\left(\sigma^{B^0}_s\right)$ for $\sigma^{B^0}_s := \mathbb{E}\left[\sqrt{\sigma^{1,1}_s}\mid B^0_{\cdot}\right]$. Thus, (4.12) implies

$$
\begin{aligned}
&\frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right)-\sigma_{2,1}\right)\mid B^0_{\cdot}\right]^2\right] ds\\
&\quad\geq \alpha^2\mathbb{E}\left[\frac{1}{t}\int_0^t \mathbb{I}_{\sigma^{B^0}_{\frac{s}{\epsilon}}\geq \tilde{h}^{-1}(\alpha+\sigma_{2,1})} ds\right]
\end{aligned}
\tag{4.13}
$$

Let now $\sigma^{\rho}_t$ be the solution to the SDE

$$
\sigma^{\rho}_t = \sigma^{B^0}_0 + \frac{1}{2}\int_0^t \left(\kappa\theta - \frac{v^2}{4}\right)\frac{1}{\sigma^{\rho}_s} ds + \frac{\kappa}{2}\int_0^t \sigma^{\rho}_s ds + \frac{\rho_2 v}{2} B^0_s
$$

The above can easily be shown to be the square root of a CIR process having the same mean-reversion and vol-of-vol as $\sigma^{1,1}$ and a different stationary mean, which however satisfies the Feller condition for not hitting zero. If for some $t_1 > 0$ we have $\sigma^{\rho}_{t_1} > \sigma^{B^0}_{t_1}$, we consider $t_0 = \inf\{s \leq t_1 : \sigma^{\rho}_s = \sigma^{B^0}_s\}$ which is obviously non-negative. Then, since $\mathbb{E}\left[\frac{1}{\sqrt{\sigma^{1,1}_s}} \mid B^0_\cdot\right] \geq \frac{1}{\mathbb{E}\left[\sqrt{\sigma^{1,1}_s} \mid B^0_\cdot\right]} = \frac{1}{\sigma^{B^0}_s}$ we have

$$\begin{aligned}
\sigma^{B^0}_{t_1} &= \sigma^{B^0}_{t_0} + \frac{1}{2}\int_0^{t_1}\left(\kappa\theta - \frac{v^2}{4}\right)\mathbb{E}\left[\frac{1}{\sqrt{\sigma^{1,1}_s}} \mid B^0_\cdot\right]ds \\
&\quad -\frac{\kappa}{2}\int_0^{t_1}\sigma^{B^0}_s ds + \frac{\rho_2 v}{2}\left(B^0_{t_1} - B^0_{t_0}\right) \\
&\geq \sigma^{B^0}_{t_0} + \frac{1}{2}\int_0^{t_1}\left(\kappa\theta - \frac{v^2}{4}\right)\frac{1}{\sigma^{B^0}_s}ds \\
&\quad -\frac{\kappa}{2}\int_0^{t_1}\sigma^{B^0}_s ds + \frac{\rho_2 v}{2}\left(B^0_{t_1} - B^0_{t_0}\right) \\
&\geq \sigma^{\rho}_{t_0} + \frac{1}{2}\int_0^{t_1}\left(\kappa\theta - \frac{v^2}{4}\right)\frac{1}{\sigma^{\rho}_s}ds \\
&\quad -\frac{\kappa}{2}\int_0^{t_1}\sigma^{B^\rho}_s ds + \frac{\rho_2 v}{2}\left(B^0_{t_1} - B^0_{t_0}\right) \\
&= \sigma^{\rho}_{t_1}
\end{aligned}$$

which is a contradiction. Thus $\sigma^{\rho}_s \leq \sigma^{B^0}_s$ for all $s \geq 0$, which can be plugged in (4.13) to give

$$\begin{aligned}
\frac{1}{t}\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left(h\left(\sigma^{1,1}_{\frac{s}{\epsilon}}\right) - \sigma_{2,1}\right) \mid B^0_\cdot\right]^2\right]ds& \\
\geq \alpha^2\mathbb{E}\left[\frac{1}{t}\int_0^t \mathbb{I}_{\sigma^{\rho}_{\frac{s}{\epsilon}} \geq \tilde{h}^{-1}(\alpha+\sigma_{2,1})}ds\right]&
\end{aligned} \tag{4.14}$$

Finally, by the positive recurrence of $\sigma^{\rho}_\cdot$ (which is the root of a CIR process, the ergodicity of which has been discussed in [9]), the RHS of the above converges to $\alpha^2\mathbb{P}\left(\sigma^{\rho,*} \geq \tilde{h}^{-1}\left(\alpha + \sigma_{2,1}\right)\right)$ as $\epsilon \to 0^+$, where $\sigma^{\rho,*}$ has the stationary distribution of $\sigma^{\rho}_\cdot$. Thus, since the square of $\sigma^{\rho}_\cdot$ satisfies Feller's boundary condition, the RHS of (4.14) converges to something strictly positive as $\epsilon \to 0^+$, which proves our claim. Of course, the above argument fails if $\rho_2 = 0$ ($\sigma^{\rho}_\cdot$ becomes deterministic), and in this case we can easily check that the LHS of (4.14) tends to zero, which implies $\tilde{\sigma} = \sigma_{2,1}$. Then we can hope for better convergence results for our model. However, $\rho_2 = 0$ means that we assume uncorrelated volatilities, while interdependence is the main feature that makes our model realistic.

*Remark* 4.3. In the case where the coefficients are non-constant but independently chosen from some distribution for each assset price, for each pair $(i,\, j)$ with $i \neq j$, the correlation between the $i$-th and the $j$-th asset prices will converge (as $\epsilon \to 0^+$) to something containing $\tilde{\sigma}_{i,j} := \sqrt{\mathbb{E}\left[h\left(\sigma^{i,j,1,*}\right)h\left(\sigma^{i,j,2,*}\right) \mid \mathcal{C}\right]}$, which will be a random quantity depending on both $i$ and $j$. However, since we assume correlated volatilities, we are unable to express this quantity as $\tilde{\sigma}_i\tilde{\sigma}_j$ for some $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$. This means that we are unable to achieve a limit of the desired form (as in (4.2), with each $X^i_\cdot$ being a logarithmicaly scaled Black-Scholes price process, driven by $W^i_\cdot$ and $W^0_\cdot$, and killed when it hits zero), since the correlation between $X^{i,*}_\cdot$ and $X^{j,*}_\cdot$ in that setting has always the form $c_ic_j$ for some $c_i, c_i \in [0, 1]$, for any $i$ and $j$ with $i \neq j$.

*Remark* 4.4. To obtain our convergence result we have assumed that the function $h$ is bounded, and that $g$ must satisfy the positive recurrence property. This means that the result we have at this stage may not be extendable to the classical Heston model, or to models that are based on CIR volatility processes which do not satisfy the conditions of Theorem 2.4. However, it covers a very wide range of stochastic volatility models which can capture many important features of large portfolios that constant volatility models cannot.

*Remark* 4.5. Even though we do not have any better approximation at this stage, which would have been great for the practical implementation of our model, the convergence result we have just proven is an important first step. Our convergence result shows also that quite accurate results can be obtained in risk management of large portfolios by using a very simple constant volatility model (like the one studied in [6], but with random coefficients), provided that the volatilities of the assets are fast mean-reverting (which is frequently observed in markets). The coefficients can be estimated by solving the corresponding stochastic boundary-initial value problem backwards, and by using regression against certain known quantities.

## 5 Fast mean-reversion - small vol-of-vol: A better model

In order to deal with the disadvantages of the model studied in the previous three sections, we will now study a different setting, where the vol-of-vol of each volatility process is allowed to be small compared to the square root of the mean-reversion coefficient. This time, the $i$-th logarithmicaly scaled asset price is assumed to evolve according to the system

$$\begin{aligned} dX^{i,\epsilon}_t &= \left(r_i - \tfrac{h^2(\sigma^{i,\epsilon}_t)}{2}\right)dt + h(\sigma^{i,\epsilon}_t)\left(\sqrt{1-\rho^2_{1,i}}dW^i_t + \rho_{1,i}dW^0_t\right), \quad 0 \leq t \leq T_i \\ d\sigma^{i,\epsilon}_t &= \tfrac{\kappa_i}{\epsilon}(\theta_i - \sigma^{i,\epsilon}_t)dt + \xi_i g\left(\sigma^{i,\epsilon}_t\right)\left(\sqrt{1-\rho^2_{2,i}}dB^i_t + \rho_{2,i}dB^0_t\right), \quad t \geq 0 \\ X^{i,\epsilon}_t &= 0, \; t > T_i := \inf\{s \geq 0 : X^{i,\epsilon}_s \leq 0\} \\ (X^{i,\epsilon}_0, \sigma^{i,\epsilon}_0) &= (x^i, \sigma^i), \end{aligned} \tag{5.1}$$

where the function $g$ is assumed to be generally continuous with at most linear growth (to include both the Ornstein-Uhlenbeck and the CIR volatility case). We also assume that $\sigma^i, \xi_i, \theta_i, \kappa_i$ are bounded random variables with $0 < \kappa_i \leq 1$ (for simplicity, since the size of the mean-reversion coefficients is measured by $\epsilon$), for every $i \in \mathbb{N}$. Under these assumptions, we will obtain a convergence result which is stronger than the one we had in the fast mean-reversion - large vol-of-vol setting, and also when the Brownian motions $W^0_\cdot$ and $B^0_\cdot$ describing the impact of the Market on each asset are allowed to be correlated. Moreover, assuming that $B^0_\cdot$ and $W^0_\cdot$ are uncorrelated and imposing better regularity on $g$, we will see that we are able to obtain a correction of order $O\left(\sqrt{\epsilon}\right)$ in a weak sense, even though this will turn out to be practically useless. The usefulness of this setting will become clear in the next section, where we will discuss the rate of convergence of probabilities of the form (1.3).

The main feature of the model we are now studying is that each volatility process $\sigma_{\cdot}^{i,\epsilon}$ converges in $L^p$ to the constant value $\theta_i$ as $\epsilon \to 0^+$, for any $p > 0$, which is a major advantage. Indeed, the reason for having very weak convergence results in the fast mean-reversion - large vol-of-vol setting was the fact that the limiting quantities $\sigma_{1,1}$, $\sigma_{2,1}$ and $\tilde{\sigma}$ did not coincide, while the corresponding limits do coincide when the volatilities converge in some strong sense to constant values, as we can easily check.

We start by establishing the convrgence of each volatility to its mean as $\epsilon \to 0^+$. This is given in the following technical lemma, the proof of which can be found in the Appendix

**Lemma 5.1.** *Suppose that $g$ is continuous and satisfies $|g(z)| \leq |z| + c_g$ for some $c_g > 0$ and for all $z \in \mathbb{R}$, and that $\sigma^i, \xi_i, \theta_i, \kappa_i$ are bounded random variables. Then, for any $t \geq 0$ and $p \geq 1$, it holds that $\sigma_s^{i,\epsilon} \to \theta_i$ in $L^p(\Omega \times [0,\, t])$, as $\epsilon \to 0^+$.*

The above is a nice convergence result, but in order to obtain a correction of some order, we need some result about the rate of convergence. This is given in the next lemma, the proof of which has also been put in the Appendix

**Lemma 5.2.** *Let $g$ be a $C^1$ function such that both $g$ and $(g^2)'$ are bounded, $h$ be an analytic function, and $\sigma^1, \xi_1, \theta_1, \kappa_1$ be bounded random variables. Moreover, suppose that the sequence $\left\{h^{(n)}(\theta_1) : n \in \mathbb{N}\right\}$ is bounded by some $M_h > 0$ for all the possible values of $\theta_1$, and that $\kappa_1 > c_\kappa > 0$ $\mathbb{P}$-almost surely, where $M_h$ and $c_\kappa$ are deterministic constants. Let also $\{Z_s : s \geq 0\}$ be a $C^1$ path such that for any $t > 0$, both $Z$ and $Z'$ are bounded in $[0,\, t]$ by some deterministic constant $M_{z,t} > 0$. Then, for any sequence $\epsilon_n \to 0^+$, there exists a subsequence $\epsilon_{k_n}$ such that for almost all $t \geq 0$ we have*

$$\begin{aligned}\frac{1}{\epsilon_{k_n}} \int_0^t \left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h(\theta_1)\right)^2 Z_s ds \quad \to \quad & \frac{Z_0}{\kappa_1} \int_{\theta_1}^{\sigma^1} \frac{\left(h(y) - h(\theta_1)\right)^2}{y - \theta_1} dy \\ & + \frac{\xi_1^2}{2\kappa_1} \left(h'(\theta_1) g(\theta_1)\right)^2 \int_0^t Z_s ds \end{aligned} \tag{5.2}$$

*and*

$$\begin{aligned}\frac{1}{\epsilon_{k_n}} \int_0^t \left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h(\theta_1)\right) Z_s ds \quad \to \quad & \frac{Z_0}{\kappa_1} \int_{\theta_1}^{\sigma^1} \frac{h(y) - h(\theta_1)}{y - \theta_1} dy \\ & + \frac{\xi_1}{\kappa_1} h'(\theta_1) g(\theta_1) \int_0^t Z_s d\tilde{B}_s^1 \\ & + \frac{\xi_1^2}{4\kappa_1} h''(\theta_1) g^2(\theta_1) \int_0^t Z_s ds \end{aligned} \tag{5.3}$$

*in $L^2(\Omega)$ as $n \to +\infty$, where $\tilde{B}_s^1$ stands for the standard Brownian Motion $\sqrt{1 - \rho_{2,1}^2} B_t^1 + \rho_{2,1} B_t^0$. If we replace the boundedness of $g$ by linear growth, the same results hold when $h$ is a polynomial.*

We proceed now to our first main result, which is the convergence of our system as $\epsilon \to 0^+$. As in previous sections, we denote by $C_i^\epsilon$ the coefficient vector of the $i$-th asset price process, by $\mathbb{E}_{\sigma,\mathcal{C}}$ the expectation given the volatility paths and the coefficient vectors, and by $L^2_{\sigma,\mathcal{C}}$ the corresponding $L^2$ norm. Then, as in the large vol-of-vol setting, we need to approximate the random mass of non-defaulted assets when the market factors are given, i.e

$$\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right) = \mathbb{E}\left[\int_0^{+\infty}\int_0^{+\infty} u_{C_1^\epsilon}\left(t,\, x,\, y\right) dxdy \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right]$$

where we have

$$u_{C_1^\epsilon}\left(t,\, x,\, y\right) = p_t^\epsilon\left(y|B_\cdot^0, \mathcal{G}\right)\mathbb{E}\left[u\left(t, x, W_\cdot^0, \mathcal{G}, C_1^\epsilon, h\left(\sigma_\cdot^{1,\epsilon}\right)\right)\,|W_\cdot^0, \sigma_t^{1,\epsilon} = y, B_\cdot^0, C_1^\epsilon, \mathcal{G}\right]$$

and where $u^\epsilon\left(t,\, x\right) := u\left(t, x, W_\cdot^0, \mathcal{G}, C_1^\epsilon, h\left(\sigma_\cdot^{1,\epsilon}\right)\right)$ solves the SPDE

$$\begin{aligned} u^\epsilon(t,\, x) \;&=\; u_0(x) - \int_0^t \left(r - \frac{h^2\left(\sigma_s^{1,\epsilon}\right)}{2}\right) u_x^\epsilon(s,\, x)ds \\ &\quad + \int_0^t \frac{h^2\left(\sigma_s^{1,\epsilon}\right)}{2} u_{xx}^\epsilon(s,\, x)ds - \rho_{1,1}\int_0^t h\left(\sigma_s^{1,\epsilon}\right) u_x^\epsilon(s,\, x)dW_s^0 \end{aligned} \tag{5.4}$$

and satisfies also the identity

$$\|u^\epsilon(t,\, \cdot)\|^2_{L^2(\mathbb{R}^+)} + \left(1 - \rho_{1,1}^2\right)\int_0^t h^2\left(\sigma_t^{1,\epsilon}\right)\|u_x^\epsilon(s,\, \cdot)\|^2_{L^2(\mathbb{R}^+)}\, ds = \|u_0\|^2_{L^2(\mathbb{R}^+)}\,. \tag{5.5}$$

for all $t \geq 0$, $\mathbb{P}$ - almost surely. Assuming again that $W_\cdot^0$ and $B_\cdot^0$ are uncorrelated and that $c < h^2\left(x\right) < C$ for some $C, c > 0$ and all $x > 0$, we can use (5.5) to show that the $L^2_\sigma$ and $L^2_\sigma\left(\Omega \times [0,\, T]\,; H_0^1\left(\mathbb{R}^+\right)\right)$ norms of $u^\epsilon$ are bounded in $\epsilon$. This implies the existence of a weak limit $u$ of $u^{\epsilon_{k_n}}$ for some subsequence $\{\epsilon_{k_n} \,:\, n \in \mathbb{N}\}$ of an arbitrary sequence $\epsilon_n \to 0^+$, in all these reflexive spaces. Then, by following exactly the same steps as in the proof of Theorem 3.1, but with $\int_0^t \sigma^{1,1}_{\frac{s}{\epsilon}}$ under the large vol-of-vol setting replaced by $\int_0^t \sigma_s^{1,\epsilon}$ under the small vol-of-vol setting, we can obtain a similar characterization for the weak limits $u$

**Theorem 5.3.** *Any weak limit $u$ of $u^\epsilon$ in $L^2(\Omega \times [0,\, T]\,; H_0^1\left(\mathbb{R}^+\right))$, in any sequence $\epsilon_n \to 0^+$ for which we have convergence, is equal to the unique solution to the SPDE*

$$\begin{aligned} u(t,\, x) \;&=\; u_0(x) - \left(r - \frac{h^2\left(\theta_1\right)}{2}\right)\int_0^t u_x(s,\, x)ds \\ &\quad + \frac{h^2\left(\theta_1\right)}{2}\int_0^t u_{xx}(s,\, x)ds - \rho_{1,1}h\left(\theta_1\right)\int_0^t u_x(s,\, x)dW_s^0 \end{aligned} \tag{5.6}$$

*in that space. Thus, the uniqueness of solutions implies that we have convergence to $u$ as $\epsilon \to 0^+$.*

Again, there are some difficulties (mainly the lack of uniform convergence of $\sigma_\cdot^{1,\epsilon}$ to $\theta_1$) which do not allow us to obtain strong convergence of $u^\epsilon$ in $L^2_\sigma(\Omega \times [0,\, T]\,; H_0^1\left(\mathbb{R}^+\right))$, while the independence between $W_\cdot^0$ and $B_\cdot^0$ is a non-realistic assumption we would like to get rid of. Fortunately, under this better setting, we can show strong convergence of the mass of $u^\epsilon$ over $\mathbb{R}^+$ in $L^2$, just by assuming that $h$ possesses a few nice properties, without the need to assume that $W^0_\cdot$ and $B^0_\cdot$ are uncorrelated. This is possible because when $h$ has these properties, we are able to show strong convergence of the antiderivative $v^{0,\epsilon} := \int_\cdot^{+\infty} u^\epsilon(y)dy$ in $L^2(\Omega \times [0, T]; H^1(\mathbb{R}^+))$. This result is given in the following Theorem

**Theorem 5.4.** *Suppose that $h$ is differentiable and that both $h$ and $h'$ have a polynomial growth. Then, $v^{0,\epsilon}$ converges strongly to $v^0$ in $L^2(\Omega \times [0, T]; H^1(\mathbb{R}^+))$ as $\epsilon \to 0^+$, for any $T > 0$, where $v^0$ is defined as $v^0(t, x) = \int_x^{+\infty} u(t, y)dy$ for all $t, x \geq 0$*

*Proof.* We can easily check that $v^{0,\epsilon}$ and $v^0$ are the unique solutions to the SPDEs (5.4) and (5.6) respectively, in the space $L^2(\Omega \times [0, T]; H^2(\mathbb{R}^+))$, under the boundary conditions $v_x^{0,\epsilon} \in L^2(\Omega \times [0, T]; H_0^1(\mathbb{R}^+))$ and $v_x^0 \in L^2(\Omega \times [0, T]; H_0^1(\mathbb{R}^+))$ respectively. Subtracting the SPDEs satisfied by $v^{0,\epsilon}$ and $v^0$ and setting $v^{d,\epsilon} = v^0 - v^{0,\epsilon}$, we can easily verify that

$$\begin{aligned} v^{d,\epsilon}(t,x) &= -\frac{1}{2}\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2(\theta_1)\right) v_x^{0,\epsilon}(s,x)\,ds \\ &\quad + \int_0^t \left(r - \frac{h^2(\theta_1)}{2}\right) v_x^{d,\epsilon}(s,x)\,ds \\ &\quad + \frac{1}{2}\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2(\theta_1)\right) v_{xx}^{0,\epsilon}(s,x)\,ds \\ &\quad + \int_0^t \frac{h^2(\theta_1)}{2} v_{xx}^{d,\epsilon}(s,x)\,ds \\ &\quad + \rho_{1,1}\int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right) - h(\theta_1)\right) v_x^{0,\epsilon}(s,x)\,dW_s^0 \\ &\quad + \rho_{1,1}\int_0^t h(\theta_1)\, v_x^{d,\epsilon}(s,x)\,dW_s^0 \end{aligned}$$

and we can use Ito's formula for the $L^2$ norm (see [21]) on the above to obtain

$$\begin{aligned} \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} \left(v^{d,\epsilon}(t,x)\right)^2 dx\right] &= -\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2(\theta_1)\right) \\ &\qquad \times \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}(s,x)\, v^{d,\epsilon}(t,x)\,dx\right] ds \\ &\quad + 2\left(r - \frac{h^2(\theta_1)}{2}\right) \\ &\qquad \times \int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} v_x^{d,\epsilon}(s,x)\, v^{d,\epsilon}(t,x)\,dx\right] ds \\ &\quad - \int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2(\theta_1)\right) \\ &\qquad \times \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}(s,x)\, v_x^{d,\epsilon}(t,x)\,dx\right] ds \\ &\quad - \int_0^t h^2(\theta_1)\, \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} \left(v_x^{d,\epsilon}(s,x)\right)^2 dx\right] ds \end{aligned}$$

$$
\begin{aligned}
&+\rho_{1,1}^2 \int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right) - h\left(\theta_1\right)\right)^2 \\
&\qquad \times \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} \left(v_x^{0,\epsilon}\left(s,\,x\right)\right)^2 dx\right] ds \\
&+2\rho_{1,1}^2 h\left(\theta_1\right) \int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right) - h\left(\theta_1\right)\right) \\
&\qquad \times \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}\left(s,\,x\right) v_x^{d,\epsilon}\left(t,\,x\right) dx\right] ds \\
&+\rho_{1,1}^2 \int_0^t h^2\left(\theta_1\right) \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} \left(v_x^{d,\epsilon}\left(s,\,x\right)\right)^2 dx\right] ds \\
&+N(t,\epsilon)
\end{aligned} \tag{5.7}
$$

where $N(t,\epsilon)$ is some noise due to the correlation between $B_\cdot^0$ and $W_\cdot^0$, with $\mathbb{E}\left[N(t,\epsilon)\right] = 0$ Next, by (5.5) we have $\left\|v_x^{0,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} = \left\|u^\epsilon(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} \leq \left\|u_0(\cdot)\right\|_{L^2(\Omega\times\mathbb{R}^+)}$ for all $s \geq 0$. Using this, we can obtain the following estimate

$$
\begin{aligned}
&\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right) \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}\left(s,\,x\right) v^{d,\epsilon}\left(t,\,x\right) dx\right] ds \\
&\quad \leq \int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right) \left\|v_x^{0,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} \left\|v^{d,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds \\
&\quad \leq \left\|u_0(\cdot)\right\|_{L^2(\Omega\times\mathbb{R}^+)} \sqrt{\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right)^2 ds} \sqrt{\int_0^t \left\|v^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds} \\
&\quad \leq \frac{1}{2} \left\|u_0(\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)} \int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right)^2 ds \\
&\qquad + \frac{1}{2} \int_0^t \left\|v^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds
\end{aligned} \tag{5.8}
$$

and in the same way

$$
\begin{aligned}
&\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right) \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}\left(s,\,x\right) v_x^{d,\epsilon}\left(t,\,x\right) dx\right] ds \\
&\quad \leq \left\|u_0(\cdot)\right\|_{L^2(\Omega\times\mathbb{R}^+)} \sqrt{\int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right)^2 ds} \sqrt{\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds} \\
&\quad \leq \frac{1}{2\eta} \left\|u_0(\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)} \int_0^t \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right)^2 ds \\
&\qquad + \frac{\eta}{2} \int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds
\end{aligned} \tag{5.9}
$$

and

$$
\begin{aligned}
&\int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right) - h\left(\theta_1\right)\right) \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} v_x^{0,\epsilon}\left(s,\,x\right) v_x^{d,\epsilon}\left(t,\,x\right) dx\right] ds \\
&\quad \leq \left\|u_0(\cdot)\right\|_{L^2(\Omega\times\mathbb{R}^+)} \sqrt{\int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right) - h\left(\theta_1\right)\right)^2 ds} \sqrt{\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds}
\end{aligned}
$$

$$\leq \frac{1}{2\eta}\left\|u_0(\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)}\int_0^t\left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)^2 ds$$
$$+\frac{\eta}{2}\int_0^t\left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds \tag{5.10}$$

for some $\eta > 0$. Moreover, we have the estimate

$$\int_0^t \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+} v_x^{d,\epsilon}(s,\, x)\, v^{d,\epsilon}(t,\, x)\, dx\right] ds$$
$$\leq \int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)}\left\|v^{d,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds$$
$$\leq \sqrt{\int_0^t \left\|v^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds}\sqrt{\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds}$$
$$\leq \frac{1}{2\eta}\int_0^t \left\|v^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds + \frac{\eta}{2}\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds \tag{5.11}$$

and by using $\left\|v_x^{0,\epsilon}(s,\cdot)\right\|_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} \leq \left\|u_0(\cdot)\right\|_{L^2(\Omega\times\mathbb{R}^+)}$ again, we also obtain

$$\int_0^t \left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)^2 \mathbb{E}_{\sigma,\mathcal{C}}\left[\int_{\mathbb{R}^+}\left(v_x^{0,\epsilon}(s,\, x)\right)^2 dx\right] ds$$
$$\leq \left\|u_0(\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)}\int_0^t\left(h\left(\sigma_s^{1,\epsilon}\right)-h\left(\theta_1\right)\right)^2 ds. \tag{5.12}$$

Plugging now (5.8), (5.9), (5.10), (5.11) and (5.12) in (5.7), and taking then $\eta$ to be sufficiently small, we get the estimate

$$\left\|v^{d,\epsilon}(t,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} + m\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds$$
$$\leq M\int_0^t \left\|v^{d,\epsilon}(s,\cdot)\right\|^2_{L^2_{\sigma,\mathcal{C}}(\Omega\times\mathbb{R}^+)} ds + N(t,\epsilon) + MH(\epsilon) \tag{5.13}$$

for all $t \in [0,\, T]$, where $H(\epsilon) = \int_0^T \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right)^2 ds + \int_0^T \left(h\left(\sigma_s^{1,\epsilon}\right) - h\left(\theta_1\right)\right)^2 ds$, and where $M, m > 0$ are constants independent of the fixed volatility path. Taking expectations on the above to average over all volatility paths, we find that

$$\left\|v^{d,\epsilon}(t,\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)} + m\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)} ds$$
$$\leq M\int_0^t \left\|v^{d,\epsilon}(s,\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)} ds + M\mathbb{E}\left[H(\epsilon)\right] \tag{5.14}$$

and using Gronwall's inequality on the above, we finally obtain

$$\left\|v^{d,\epsilon}(t,\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)} + m\int_0^t \left\|v_x^{d,\epsilon}(s,\cdot)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)} ds \leq M'\mathbb{E}\left[H(\epsilon)\right] \tag{5.15}$$

for some $M' > 0$, with $\mathbb{E}[H(\epsilon)] \to 0^+$ as $\epsilon \to 0^+$ as we can easily show. Indeed, if $h(x) \le x^m + c_{h,1}$ and $h'(x) \le x^m + c_{h,2}$ for all $x \ge 0$, by the mean value theorem we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left(h\left(\sigma_s^{1,\epsilon}\right) - h\left(\theta_1\right)\right)^2 ds\right] &= \mathbb{E}\left[\int_0^T h'\left(\sigma_s^{*,\epsilon}\right)\left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 ds\right] \\
&\le c_{h,2}\mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 ds\right] \\
&\quad +\mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 \left(\sigma_s^{1,\epsilon}\right)^m ds\right] \\
&\le c_{h,2}\mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 ds\right] \\
&\quad +\sqrt{\mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon}\right)^{2m} ds\right]} \\
&\qquad \times\sqrt{\mathbb{E}\left[\int_0^T \left(\sigma_s^{1,\epsilon} - \theta_1\right)^4 ds\right]}
\end{aligned}$$

with the RHS of the last tending to zero by Lemma 5.1, and in the same way we can show that $\mathbb{E}\left[\int_0^T \left(h^2\left(\sigma_s^{1,\epsilon}\right) - h^2\left(\theta_1\right)\right)^2 ds\right]$ tends also to zero (since $\left(h^2\right)' = 2hh'$ has also a polynomial growth). The proof of the Theorem is now complete. □

*Remark* 5.5. By Morrey's inequality in dimension 1 (see [8]) and the above result, we can easily obtain the convergence of $v^{0,\epsilon}(\cdot, 0)$ to $v^0(\cdot, 0)$ in $L^2(\Omega \times [0, T])$ as $\epsilon \to 0^+$. Taking expectations given $\left(W_\cdot^0, B_\cdot^0, \mathcal{G}\right)$ (i.e averaging over the volatility path and the vector $C_1'$, we get the strong convergence result for the masses of non-defaulted assets. Also, under the appropriate regularity conditions on $h$ and $g$, by Lemma 5.2 we have that $\mathbb{E}[H(\epsilon)] = \mathcal{O}(\epsilon)$, so (5.15) implies that the rate of the above convergence is $\sqrt{\epsilon}$.

We continue now with the study of the existence of a correction of some order for $u^\epsilon$. For this purpose, we study the asymptotic behaviour of $v^{1,\epsilon} := \frac{v^{d,\epsilon}}{\sqrt{\epsilon}} = \frac{v^{0,\epsilon} - v^0}{\sqrt{\epsilon}}$. Dividing the estimate (5.15) in the previous proof by $\epsilon$, since Lemma 5.2 implies that $\mathbb{E}[H(\epsilon)] = \mathcal{O}(\epsilon)$ (under the appropriate regularity conditions on $h$ and $g$), we find that

$$\left\|v^{1,\epsilon}(t, \cdot)\right\|^2_{L^2(\Omega \times \mathbb{R}^+)} + m\int_0^t \left\|v_x^{1,\epsilon}(s, \cdot)\right\|^2_{L^2(\Omega \times \mathbb{R}^+)} ds \le M'' \tag{5.16}$$

for some $M'' > 0$ and all $t, \epsilon > 0$. This implies that for any sequence $\{\epsilon_n : n \in \mathbb{N}\}$, there exists a subsequence $\{\epsilon_{k_n} : n \in \mathbb{N}\}$ such that $v^{1,\epsilon_{k_n}}$ converges weakly to some $v^1$ in $L^2([0, T] \times \Omega; H^1(\mathbb{R}^+))$ as $n \to +\infty$, for any $T > 0$. Below we will show that the weak limit $v^1$ is always a solution to an SPDE, but since we will not prove convergence of second order derivatives, the Neumann boundary condition satisfied by $v^{1,\epsilon_{k_n}}$ will not be established in the limit. This means that we are currently unable to show uniqueness of weak solutions to the limiting SPDE, and thus uniqueness of weak limits $v^1$. Therefore, we see that even in this small vol-of-vol setting, at this point we cannot hope for good high order corrections, like the ones established in [9] for the prices of vanilla options. Moreover, to prove the above weak convergence result, we need to assume again that $W_\cdot^0$ and $B_\cdot^0$ are uncorrelated, which is not a reallistic assumption as we have already mentioned. However, we will see in the next section that under certain circumstances, by using 5.5 we are able to estimate the exact rate of convergence of probabilities of the form (1.3), even when $W^0_\cdot$ and $B^0_\cdot$ are correlated, something we couldn't do in the fast mean-reversion - large vol-of-vol setting. We close this section by proving the Theorem which characterizes the weak limits $v^1$

**Theorem 5.6.** *Suppose that $g$ is a $C^1$ function such that $\left(g^2\right)'$ is bounded, and that $h$ is an analytic function such that both $\left\{h^{(n)}\left(\theta_1\right) : n \in \mathbb{N}\right\}$ and $\left\{\left(h^2\right)^{(n)}\left(\theta_1\right) : n \in \mathbb{N}\right\}$ are bounded by a deterministic constant. Suppose also that either $g$ is bounded or $h$ is a polynomial, and that $W^0_\cdot$ and $B^0_\cdot$ are uncorrelated. Then, the weak limit $v^1$ of $v^{1,\epsilon_{k_n}}$ in $L^2([0,\,T]\times\Omega;H^1\left(\mathbb{R}^+\right))$ is a weak solution to the SPDE*

$$\begin{aligned}
v^1\left(t,\,x\right) &= -\int_0^t\left(r-\frac{h^2\left(\theta_1\right)}{2}\right)v^1_x\left(s,\,x\right)ds\\
&\quad+\int_0^t\frac{h^2\left(\theta_1\right)}{2}v^1_{xx}\left(s,\,x\right)ds\\
&\quad-\int_0^t\rho_{1,1}h\left(\theta_1\right)v^1_x\left(s,\,x\right)dW^0_s
\end{aligned}$$

*in that space.*

*Proof.* The regularity we have assumed here allows us to apply Lemma 5.2 whenever we need it (including the proof of estimate (5.16) earlier, which gives the existence of weak limits). As in the proof of Theorem 5.4, we have that $v^{d,\epsilon_{k_n}} = v^{0,\epsilon_{k_n}} - v^0$ satisfies

$$\begin{aligned}
v^{d,\epsilon_{k_n}}\left(t,\,x\right) &= -\frac{1}{2}\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h^2\left(\theta_1\right)\right)v_x^{0,\epsilon_{k_n}}\left(s,\,x\right)ds\\
&\quad+\int_0^t\left(r-\frac{h^2\left(\theta_1\right)}{2}\right)v_x^{d,\epsilon_{k_n}}\left(s,\,x\right)ds\\
&\quad+\frac{1}{2}\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h^2\left(\theta_1\right)\right)v_{xx}^{0,\epsilon_{k_n}}\left(s,\,x\right)ds\\
&\quad+\int_0^t\frac{h^2\left(\theta_1\right)}{2}v_{xx}^{d,\epsilon_{k_n}}\left(s,\,x\right)ds\\
&\quad+\rho_{1,1}\int_0^t\left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h\left(\theta_1\right)\right)v_x^{0,\epsilon_{k_n}}\left(s,\,x\right)dW_s^0\\
&\quad+\rho_{1,1}\int_0^t h\left(\theta_1\right)v_x^{d,\epsilon_{k_n}}\left(s,\,x\right)dW_s^0
\end{aligned}$$

so dividing by $\sqrt{\epsilon_{k_n}}$ we find

$$\begin{aligned}
v^{1,\epsilon_{k_n}}\left(t,\,x\right) &= -\frac{1}{2\sqrt{\epsilon_{k_n}}}\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h^2\left(\theta_1\right)\right)v_x^{0,\epsilon_{k_n}}\left(s,\,x\right)ds\\
&\quad+\int_0^t\left(r-\frac{h^2\left(\theta_1\right)}{2}\right)v_x^{1,\epsilon_{k_n}}\left(s,\,x\right)ds\\
&\quad+\frac{1}{2\sqrt{\epsilon_{k_n}}}\int_0^t\left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right)-h^2\left(\theta_1\right)\right)v_{xx}^{0,\epsilon_{k_n}}\left(s,\,x\right)ds
\end{aligned}$$

$$+\int_0^t \frac{h^2(\theta_1)}{2} v_{xx}^{1,\epsilon_{k_n}}(s,\, x)\, ds$$
$$+\frac{\rho_{1,1}}{\sqrt{\epsilon_{k_n}}} \int_0^t \left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h(\theta_1)\right) v_x^{0,\epsilon_{k_n}}(s,\, x)\, dW_s^0$$
$$+\rho_{1,1} \int_0^t h(\theta_1)\, v_x^{1,\epsilon_{k_n}}(s,\, x)\, dW_s^0 \tag{5.17}$$

for any $t \in [0,\, T]$. We test now the above against a smooth and compactly supported function $f$, and a $W_\cdot^0$-measurable random variable $Z \in L^2(\Omega)$, with $Z = \mathbb{E}[Z] + \int_0^T z_s dW_s^0$ for some smooth process $z_\cdot$ whose derivatives have bounded moments in $[0,\, T]$. This way we obtain

$$\begin{aligned}
&\mathbb{E}_\sigma\left[Z \int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}(t,\, x)\, f(x) dx\right] \\
&\quad = \frac{1}{2\sqrt{\epsilon_{k_n}}} \int_0^t \left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h^2(\theta_1)\right) \mathbb{E}_\sigma\left[Z \int_{\mathbb{R}^+} v^{0,\epsilon_{k_n}}(s,\, x)\, f'(x) dx\right] ds \\
&\qquad - \int_0^t \left(r - \frac{h^2(\theta_1)}{2}\right) \mathbb{E}_\sigma\left[Z \int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}(s,\, x)\, f'(x) dx\right] ds \\
&\qquad + \frac{1}{2\sqrt{\epsilon_{k_n}}} \int_0^t \left(h^2\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h^2(\theta_1)\right) \mathbb{E}_\sigma\left[Z \int_{\mathbb{R}^+} v^{0,\epsilon_{k_n}}(s,\, x)\, f''(x) dx\right] ds \\
&\qquad + \int_0^t \frac{h^2(\theta_1)}{2} \mathbb{E}_\sigma\left[Z \int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}(s,\, x)\, f''(x) dx\right] ds \\
&\qquad - \frac{\rho_{1,1}}{\sqrt{\epsilon_{k_n}}} \int_0^t \left(h\left(\sigma_s^{1,\epsilon_{k_n}}\right) - h(\theta_1)\right) \mathbb{E}_\sigma\left[z_s \int_{\mathbb{R}^+} v^{0,\epsilon_{k_n}}(s,\, x)\, f'(x) dx\right] ds \\
&\qquad - \rho_{1,1} \int_0^t h(\theta_1)\, \mathbb{E}_\sigma\left[z_s \int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}(s,\, x)\, f'(x) dx\right] ds
\end{aligned} \tag{5.18}$$

for any $t \in [0,\, T]$.

Observe now that for $g \in \{h,\, h^2\}$, $V_\cdot \in \{Z, z_\cdot\}$ and $k \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\frac{1}{\sqrt{\epsilon_{k_n}}} \int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g(\theta_1)\right) \mathbb{E}_\sigma\left[V_s \int_{\mathbb{R}^+} v^{0,\epsilon_{k_n}}(s,\, x)\, f^{(k)}(x) dx\right] ds \\
&\quad = \int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g(\theta_1)\right) \mathbb{E}_\sigma\left[V_s \int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}(s,\, x)\, f^{(k)}(x) dx\right] ds \\
&\qquad + \frac{1}{\sqrt{\epsilon_{k_n}}} \int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g(\theta_1)\right) \mathbb{E}_\sigma\left[V_s \int_{\mathbb{R}^+} v^{0}(s,\, x)\, f^{(k)}(x) dx\right] ds
\end{aligned} \tag{5.19}$$

where for any $\left(B_\cdot^0,\, B_\cdot^1\right)$ - measurable random variable $U$ which is bounded by some $M_U > 0$, by using standard norm estimates and (5.16) we have

$$\begin{aligned}
&\left|\mathbb{E}\left[U \int_0^t \left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right) - g(\theta_1)\right) \mathbb{E}_\sigma\left[V_s \int_{\mathbb{R}^+} v^{1,\epsilon_{k_n}}(s,\, x)\, f^{(k)}(x) dx\right] ds\right]\right| \\
&\quad \leq \left\|f^{(k)}\right\|_{L^2(\mathbb{R}^+)} \sup_{0 \leq s \leq T} \|V_s\|_{L^2(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times\mathbb{E}\left[\int_0^t |U|\left|g\left(\sigma_s^{1,\epsilon_{k_n}}\right)-g\left(\theta_1\right)\right|\left\|v^{1,\epsilon_{k_n}}\left(s,\cdot\right)\right\|_{L^2_\sigma(\Omega\times\mathbb{R}^+)}ds\right]\\
&\leq \left\|f^{(k)}\right\|_{L^2(\mathbb{R}^+)}\sup_{0\leq s\leq T}\|V_s\|_{L^2(\Omega)}\\
&\quad \times M_U\sqrt{\mathbb{E}\left[\int_0^t\left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right)-g\left(\theta_1\right)\right)^2ds\right]}\sqrt{\int_0^t\left\|v^{1,\epsilon_{k_n}}\left(s,\cdot\right)\right\|^2_{L^2(\Omega\times\mathbb{R}^+)}ds}\\
&\leq \sqrt{tM''}M_U\left\|f^{(k)}\right\|_{L^2(\mathbb{R}^+)}\sup_{0\leq s\leq T}\|V_s\|_{L^2(\Omega)}\sqrt{\mathbb{E}\left[\int_0^t\left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right)-g\left(\theta_1\right)\right)^2ds\right]}
\end{aligned}
$$

which tends to zero by Lemma 5.1, in a subsequence $\{\epsilon_{k'_n} : n \in \mathbb{N}\}$ of $\{\epsilon_{k_n} : n \in \mathbb{N}\}$. Thus, since bounded random variables are dense in $L^2(\Omega)$, we have that the first term in the RHS of (5.19) tends to zero weakly in $L^2(\Omega)$ (where we average over all the volatility paths) and for almost all $t > 0$. Moreover, since $v^0$ is bounded by 1 we have

$$
\mathbb{E}_\sigma\left[V_s\int_{\mathbb{R}^+}v^0\left(s,\,x\right)f^{(k)}(x)dx\right]\leq\left\|f^{(k)}\right\|_{L^1(\mathbb{R}^+)}\sup_{0\leq s\leq T}\|V_s\|_{L^1(\Omega)}
$$

for any $s \in [0,\, T]$, while we can test the SPDE satisfied by $v^0$ against $f^{(k)}$ and $V_\cdot$ to show that the derivative of the LHS of the above is a linear combination of terms of the same form, which means that this derivative is also bounded. Thus, by Lemma 5.2 we have that the second term in the RHS of (5.19), i.e

$$
\begin{aligned}
&\frac{1}{\sqrt{\epsilon_{k_n}}}\int_0^t\left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right)-g\left(\theta_1\right)\right)\mathbb{E}_\sigma\left[V_s\int_{\mathbb{R}^+}v^0\left(s,\,x\right)f^{(k)}(x)dx\right]ds\\
&\quad=\sqrt{\epsilon_{k_n}}\frac{1}{\epsilon_{k_n}}\int_0^t\left(g\left(\sigma_s^{1,\epsilon_{k_n}}\right)-g\left(\theta_1\right)\right)\mathbb{E}_\sigma\left[V_s\int_{\mathbb{R}^+}v^0\left(s,\,x\right)f^{(k)}(x)dx\right]ds,
\end{aligned}
$$

tends also to zero in $L^2(\Omega)$, for almost all $t > 0$, in a further subsequence $\{\epsilon_{k''_n} : n \in \mathbb{N}\}$ of $\{\epsilon_{k'_n} : n \in \mathbb{N}\}$. Therefore, in this subsequence, the LHS of (5.19) tends to zero weakly in $L^2(\Omega)$, for almost all $t > 0$.

Next, by our weak convergence result, for any $g \in \{h,\, h^2\}$, $V_\cdot \in \{Z, z_\cdot\}$ and $k \in \mathbb{N}$, it holds that

$$
\begin{aligned}
&\int_0^t g\left(\theta_1\right)\mathbb{E}_\sigma\left[V_s\int_{\mathbb{R}^+}v^{1,\epsilon_{k_n}}\left(s,\,x\right)f^{(k)}(x)dx\right]ds\\
&\qquad\longrightarrow\int_0^t g\left(\theta_1\right)\mathbb{E}_\sigma\left[V_s\int_{\mathbb{R}^+}v^1\left(s,\,x\right)f^{(k)}(x)dx\right]ds
\end{aligned}
$$

weakly in $L^2(\Omega)$ and for all $t \in [0,\, T]$, as $n \to +\infty$. Thus, by recalling the convergence of the LHS of (5.19) to zero as well, we find that the RHS of (5.18) converges weakly in $L^2(\Omega)$ and for all $t \in [0,\, T]$ to

$$
\begin{aligned}
R(t)\;&=\;-\int_0^t\left(r-\frac{h^2\left(\theta_1\right)}{2}\right)\mathbb{E}_\sigma\left[Z\int_{\mathbb{R}^+}v^1\left(s,\,x\right)f'(x)dx\right]ds\\
&\quad+\int_0^t\frac{h^2\left(\theta_1\right)}{2}\mathbb{E}_\sigma\left[Z\int_{\mathbb{R}^+}v^1\left(s,\,x\right)f''(x)dx\right]ds
\end{aligned}
$$

$$-\rho_{1,1}\int_0^t h(\theta_1)\,\mathbb{E}_\sigma\left[z_s\int_{\mathbb{R}^+}v^1(s,\,x)\,f'(x)dx\right]ds \tag{5.20}$$

in the subsequence $\{\epsilon_{k''_n} : n\in\mathbb{N}\}$ of $\{\epsilon_{k_n} : n\in\mathbb{N}\}$.

We will prove now that the above convergence holds also weakly in $L^2(\Omega\times[0,\,T])$. If we test all the terms in the RHS of (5.18) against some $\left(B^0_\cdot,\,B^1_\cdot\right)$ - measurable random variable $U\in L^2(\Omega)$, we have that the occurring quantities converge in the subsequence $\{\epsilon_{k''_n} : n\in\mathbb{N}\}$ for almost all $t\in[0,\,T]$, and we need to show that this convergence holds also weakly in $L^2([0,\,T])$. This can be shown easily by using the Dominated Convergence Theorem, since we can show that these quantities are uniformly bounded in $t\in[0,\,T]$. Indeed, since $v^{0,\epsilon_{k_n}}$ is bounded by 1, by using the Cauchy-Schwartz inequality and other standard norm estimates we have

$$\begin{aligned}
&\mathbb{E}\left[U\frac{1}{\sqrt{\epsilon_{k''_n}}}\int_0^t\left(g\left(\sigma_s^{1,\epsilon_{k''_n}}\right)-g(\theta_1)\right)\mathbb{E}_\sigma\left[V_s\int_{\mathbb{R}^+}v^{0,\epsilon_{k''_n}}(s,\,x)\,f^{(k)}(x)dx\right]ds\right]\\
&\quad\le\left\|f^{(k)}\right\|_{L^1(\mathbb{R}^+)}\sup_{0\le s\le T}\|V_s\|_{L^1(\Omega)}\,\mathbb{E}\left[|U|\frac{1}{\sqrt{\epsilon_{k''_n}}}\int_0^t\left|g\left(\sigma_s^{1,\epsilon_{k''_n}}\right)-g(\theta_1)\right|ds\right]\\
&\quad\le\left\|f^{(k)}\right\|_{L^1(\mathbb{R}^+)}\sup_{0\le s\le T}\|V_s\|_{L^1(\Omega)}\,\|U\|_{L^2(\Omega)}\,\mathbb{E}\left[\frac{T}{\epsilon_{k''_n}}\int_0^T\left(g\left(\sigma_s^{1,\epsilon_{k''_n}}\right)-g(\theta_1)\right)^2ds\right]
\end{aligned}$$

and by using also estimate (5.16)

$$\begin{aligned}
&\mathbb{E}\left[U\int_0^t g(\theta_1)\,\mathbb{E}_\sigma\left[V_s\int_{\mathbb{R}^+}v^{1,\epsilon_{k_n}}(s,\,x)\,f^{(k)}(x)dx\right]ds\right]\\
&\qquad\le Tg(\theta_1)\,M''\left\|f^{(k)}\right\|_{L^2(\mathbb{R}^+)}\sup_{0\le s\le T}\|V_s\|_{L^1(\Omega)}\,\|U\|_{L^1(\Omega)}
\end{aligned}$$

for any $g\in\{h,\,h^2\}$, $V_\cdot\in\{Z,z_\cdot\}$ and $k\in\mathbb{N}$, with the RHS of the first being independent of $t$ and convergent as $n\to+\infty$ (by Lemma 5.2), and the RHS of the second being a nice uniform bound.

Finally, since we have shown that the RHS of (5.18) converges (in a subsequence of $\{\epsilon_{k_n} : n\in\mathbb{N}\}$) to $R(\cdot)$, weakly in $L^2(\Omega\times[0,\,T])$, and since the LHS of (5.18) converges in the same topology to

$$\mathbb{E}_\sigma\left[Z\int_{\mathbb{R}^+}v^1(t,\,x)\,f(x)dx\right],$$

by the uniqueness of weak limits and (5.20) we must have

$$\begin{aligned}
\mathbb{E}_\sigma\left[Z\int_{\mathbb{R}^+}v^1(t,\,x)\,f(x)dx\right] &= -\int_0^t\left(r-\frac{h^2(\theta_1)}{2}\right)\mathbb{E}_\sigma\left[Z\int_{\mathbb{R}^+}v^1(s,\,x)\,f'(x)dx\right]ds\\
&\quad+\int_0^t\frac{h^2(\theta_1)}{2}\mathbb{E}_\sigma\left[Z\int_{\mathbb{R}^+}v^1(s,\,x)\,f''(x)dx\right]ds\\
&\quad-\rho_{1,1}\int_0^t h(\theta_1)\,\mathbb{E}_\sigma\left[z_s\int_{\mathbb{R}^+}v^1(s,\,x)\,f'(x)dx\right]ds\\
&= -\mathbb{E}_\sigma\left[Z\int_0^t\left(r-\frac{h^2(\theta_1)}{2}\right)\int_{\mathbb{R}^+}v^1(s,\,x)\,f'(x)dxds\right]
\end{aligned}$$

$$
+\mathbb{E}_{\sigma}\left[Z\int_{0}^{t}\frac{h^{2}\left(\theta_{1}\right)}{2}\int_{\mathbb{R}^{+}}v^{1}\left(s,\,x\right)f''(x)dxds\right]
$$
$$
-\mathbb{E}_{\sigma}\left[Z\int_{0}^{t}\rho_{1,1}h\left(\theta_{1}\right)\int_{\mathbb{R}^{+}}v^{1}\left(s,\,x\right)f'(x)dxdW_{s}^{0}\right].
$$

The desired result follows, since the space of random variables $Z=\mathbb{E}\left[Z\right]+\int_{0}^{T}z_{s}dW_{s}^{0}$ for which $z_{\cdot}$ has derivatives with bounded moments in $[0,\,T]$ is a dense subspace of $L_{\sigma}^{2}\left(\Omega\right)$, no matter what the volatility paths are. $\square$

# 6 Fast mean-reversion - small vol-of-vol: discussion of the rate of convergence

What we have now is the strong convergence of our system as $\epsilon\rightarrow 0^{+}$, and a weak result for characterizing a possible correction of order $\mathcal{O}\left(\sqrt{\epsilon}\right)$. While the last weak result seems to be the strongest possible, it doesn't seem to be really useful, mainly due to the lack of uniqueness of solutions to the SPDE which characterizes possible corrections. It seems thus that both in this small vol-of-vol setting and the fast mean-reversion - large vol-of-vol setting studied in earlier sections, we can only have convergence of probabilities of the form (1.3), which means that the two settings are equally good. However, we will see that this is not the case, since in the large vol-of-vol setting we have shown that we have only weak convergence of default masses, while in this small vol-of-vol setting we have established strong convergence, which will allow us to estimate the exact rate of convergence of probabilities of the form (1.3). When a certain regularity condition is satisfied at both $a$ and $b$, this rate is going to be of order $\mathcal{O}\left(\sqrt[3]{\epsilon}\right)$.

To compute the rate of convergence mentioned above, we define first the limiting model, where the $i$-th logarithmicaly scaled asset price $X_{\cdot}^{i,*}$ evolves in time according to

$$
\begin{aligned}
dX_{t}^{i,*} &= \left(r_{i}-\tfrac{h^{2}(\theta_{i})}{2}\right)dt+h(\theta_{i})\left(\sqrt{1-\rho_{1,i}^{2}}dW_{t}^{i}+\rho_{1,i}dW_{t}^{0}\right),\;\; 0\leq t\leq T_{i} \\
X_{t}^{i,*} &= 0,\; t>T_{i}:=\inf\{s\geq 0: X_{s}^{i,*}\leq 0\} \\
X_{0}^{i,*} &= x^{i}
\end{aligned}
\tag{6.1}
$$

for all $i\in\mathbb{N}$. Under this model, the random mass of non-defaulted assets equals $\mathbb{P}\left(X_{t}^{1,*}>0\,|\,W_{\cdot}^{0},\,\mathcal{G}\right)$, with $\mathbb{P}\left(X_{t}^{1,*}>0\,|\,W_{\cdot}^{0},\,C_{1}',\,\mathcal{G}\right)=v^{0}\left(t,0\right)=\int_{0}^{+\infty}u\left(t,x\right)dx$, where $u$ is the unique solution to the SPDE (5.6) in $L^{2}(\Omega\times[0,\,T]\,;H_{0}^{1}\left(\mathbb{R}^{+}\right))$. We consider now the approximation error for the probabilities we are approximating, i.e

$$
\begin{aligned}
&E(x,T) \\
&\quad=\int_{0}^{T}\left|\mathbb{P}\left(\mathbb{P}\left(X_{t}^{1,\epsilon}>0\,|\,W_{\cdot}^{0},\,B_{\cdot}^{0},\,\mathcal{G}\right)>x\right)-\mathbb{P}\left(\mathbb{P}\left(X_{t}^{1,*}>0\,|\,W_{\cdot}^{0},\,\mathcal{G}\right)>x\right)\right|dt
\end{aligned}
$$

for $x\in[0,\,1]$, and we will show that this error is expected to be of order $\mathcal{O}\left(\sqrt[3]{\epsilon}\right)$, in the worst case. Indeed, let $\mathcal{E}_{t,\,\epsilon}=\{\omega\in\Omega:\,\mathbb{P}\left(X_{t}^{1,\epsilon}>0\,|\,W_{\cdot}^{0},\,B_{\cdot}^{0},\,\mathcal{G}\right)>x\}$ for $\epsilon>0$, $\mathcal{E}_{t,\,0}=\{\omega\in\Omega:\,\mathbb{P}\left(X_{t}^{1,*}>0\,|\,W_{\cdot}^{0},\,\mathcal{G}\right)>x\}$, and observe that

$$
E(x,T)\;=\;\int_{0}^{T}\left|\mathbb{P}\left(\mathcal{E}_{t,\,\epsilon}\right)-\mathbb{P}\left(\mathcal{E}_{t,\,0}\right)\right|dt,
$$

$$
\begin{aligned}
&= \int_0^T \left|\mathbb{P}\left(\mathcal{E}_{t,\epsilon} \cap \mathcal{E}_{t,0}^c\right) - \mathbb{P}\left(\mathcal{E}_{t,0} \cap \mathcal{E}_{t,\epsilon}^c\right)\right| dt, \\
&\leq \int_0^T \mathbb{P}\left(\mathcal{E}_{t,\epsilon} \cap \mathcal{E}_{t,0}^c\right) dt + \int_0^T \mathbb{P}\left(\mathcal{E}_{t,0} \cap \mathcal{E}_{t,\epsilon}^c\right) dt,
\end{aligned}
$$

Next, for any $\eta > 0$ we have

$$
\begin{aligned}
&\mathbb{P}\left(\mathcal{E}_{t,\epsilon} \cap \mathcal{E}_{t,0}^c\right) \\
&\quad = \mathbb{P}\left(\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right) > x > \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right)\right) \\
&\quad = \mathbb{P}\left(\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right) > x > x - \eta > \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right)\right) \\
&\qquad + \mathbb{P}\left(\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right) > x > \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right) > x - \eta\right) \\
&\quad \leq \mathbb{P}\left(\left|\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right) - \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right)\right| > \eta\right) \\
&\qquad + \mathbb{P}\left(x > \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right) > x - \eta\right) \\
&\quad \leq \frac{1}{\eta^2}\mathbb{E}\left[\left(\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right) - \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right)\right)^2\right] \\
&\qquad + \mathbb{P}\left(x > \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right) > x - \eta\right) \qquad (6.2)
\end{aligned}
$$

and if we denote by $\mathcal{S}$ the $\sigma$-algebra generated by the volatility paths, since $X_t^{1,*}$ is independent of $\mathcal{S}$ and the path of $B_\cdot^0$, by Cauchy-Schwartz inequality we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left(\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\right) - \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right)\right)^2\right] \\
&\quad = \mathbb{E}\Big[\Big(\mathbb{E}\Big[\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, C_1',\, \mathcal{S},\, \mathcal{G}\right) \\
&\qquad\quad - \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, C_1',\, \mathcal{G}\right) \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\Big]\Big)^2\Big] \\
&\quad \leq \mathbb{E}\Big[\mathbb{E}\Big(\Big[\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, C_1',\, \mathcal{S},\, \mathcal{G}\right) \\
&\qquad\quad - \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, C_1',\, \mathcal{G}\right)\Big)^2 \,|\, W_\cdot^0,\, B_\cdot^0,\, \mathcal{G}\Big]\Big] \\
&\quad = \mathbb{E}\left[\left(\mathbb{P}\left(X_t^{1,\epsilon} > 0 \,|\, W_\cdot^0,\, C_1',\, \mathcal{S},\, \mathcal{G}\right) - \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, C_1',\, \mathcal{G}\right)\right)^2\right] \\
&\quad = \left\|v^{0,\epsilon}(t,0) - v^0(t,0)\right\|_{L^2(\Omega)}^2 . \qquad (6.3)
\end{aligned}
$$

We assume now that $\mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right)$ has a bounded density near $x$. This is something we are not going to prove here, but we expect it to hold for almost all (if not all) $x$. Then we have that

$$
\mathbb{P}\left(x > \mathbb{P}\left(X_t^{1,*} > 0 \,|\, W_\cdot^0,\, \mathcal{G}\right) > x - \eta\right) = \mathcal{O}(\eta). \qquad (6.4)
$$

Thus, we can plug (6.3) and (6.4) in (6.2) to obtain

$$
\mathbb{P}\left(\mathcal{E}_{t,\epsilon} \cap \mathcal{E}_{t,0}^c\right) \leq \left\|v^{0,\epsilon}(t,0) - v^0(t,0)\right\|_{L^2(\Omega)}^2 + \mathcal{O}(\eta) \qquad (6.5)
$$

for any $\eta > 0$, and in a similar way we can obtain

$$\mathbb{P}\left(\mathcal{E}_{t,0} \cap \mathcal{E}_{t,\epsilon}^c\right) \leq \left\| v^{0,\epsilon}(t,0) - v^0(t,0) \right\|_{L^2(\Omega)}^2 + \mathcal{O}(\eta) \tag{6.6}$$

Finally, plugging the above two estimates in (6.2), taking $\eta = \epsilon^p$ for some $p > 0$, and recalling Remark 5.5, we find that

$$E(x,T) \leq \mathcal{O}(\epsilon^p) + \mathcal{O}\left(\epsilon^{1-2p}\right) \tag{6.7}$$

which becomes optimal as $\epsilon \to 0^+$ when $1-2p = p \Leftrightarrow p = \frac{1}{3}$. This gives $E(x,T) = \mathcal{O}(\sqrt[3]{\epsilon})$ as $\epsilon \to 0^+$.

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# A APPENDIX: Proofs of technical results

In this Appendix we prove Theorem 2.3, Theorem 2.4, and the two technical lemmas from Section 5.

**Proof of Theorem 2.3.** It suffices to show that the two-dimensional continuous Markov chain $\left(\sigma_\cdot^{1,1},\, \sigma_\cdot^{2,1}\right)$ is positive recurrent. To do this, we set $H^i(x) = \int_0^x \frac{1}{v_i g(y)} dy$ which is a strictly increasing bijection from $\mathbb{R}$ to itself, and $Z_\cdot^i = H^i\left(\sigma_\cdot^{i,1}\right)$ for $i \in \{1,\, 2\}$. Then we need to show that $\left(Z_\cdot^1,\, Z_\cdot^2\right)$ is a positive recurrent diffusion. It is easy to verify now that the infinitesimal generator $L_Z$ of $Z = \left(Z_\cdot^1,\, Z_\cdot^2\right)$, maps any smooth function $F : \mathbb{R}^2 \to \mathbb{R}$ to

$$L_Z F(x,\, y) = V^1(x) F_x(x,\, y) + V^2(y) F_y(x,\, y) + \frac{1}{2}\left(F_{xx}(x,\, y) + F_{yy}(x,\, y)\right) + \lambda F_{xy}(x,\, y)$$

for $\lambda = \rho_{2,1}\rho_{2,2} < 1$ and $V^i(x) = \frac{\kappa_i\left(\theta_i - \left(H^i\right)^{-1}(x)\right)}{v_i g\left(\left(H^i\right)^{-1}(x)\right)} - \frac{v_i}{2} g'\left(\left(H^i\right)^{-1}(x)\right)$ for $i \in \{1,\, 2\}$, which are two continuous and strictly decreasing bijections from $\mathbb{R}$ to itself. We shall use now Theorem 2.5 from [27]. Under the notation of that paper, we can easily compute

$$\begin{aligned} A_{(z,\,w)}(s,\, (x,\, y)) &= \frac{1}{2} + \lambda \frac{(x-z)(y-w)}{(x-z)^2 + (y-w)^2} \\ &\geq \frac{1}{2} + \lambda \frac{-\frac{1}{2}\left((x-z)^2 + (y-w)^2\right)}{(x-z)^2 + (y-w)^2} \\ &= \frac{1}{2}(1-\lambda) > 0 \end{aligned} \tag{A.1}$$

and also $B(s,\,(x,\,y)) = 1$ and

$$C_{(z,\,w)}(s,\,(x,\,y)) = 2\left(V^1(x)(x-z) + V^2(y)(y-w)\right) \tag{A.2}$$

for all $(x,\,y),(z,\,w) \in \mathbb{R}^2$. Since the coefficients of $L_Z$ are continuous, with the higher order ones being constant, we can easily verify condition $A_1 - A_2$ from [27]. Moreover, since $B$ and $C_{(z,\,w)}$ are continuous and $A_{(z,\,w)}$ lower-bounded by $\frac{1}{2}(1-\lambda) > 0$, we have $A_5$ as well. Next, we choose $z$ and $w$ to be the unique roots of $V^1(x)$ and $V^2(y)$ respectively, and with the notation of [27] we have

$$\underline{\alpha}(r;(z,\,w),0) = \inf_{(x-z)^2+(y-w)^2=r^2} A_{(z,\,w)}(s,\,(x,\,y)) \geq \frac{1}{2}(1-\lambda) > 0 \tag{A.3}$$

and

$$\begin{aligned}
\overline{\beta}(r;(z,\,w),0) &= \sup_{(x-z)^2+(y-w)^2=r^2} \frac{B(s,\,(x,\,y)) - A_{(z,\,w)}(s,\,(x,\,y)) + C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))} \\
&\leq \frac{2}{1-\lambda} - 1 + \frac{2}{1+\lambda} \sup_{(x-z)^2+(y-w)^2=r^2} C_{(z,\,w)}(s,\,(x,\,y))
\end{aligned} \tag{A.4}$$

since $C_{(z,\,w)}(s,\,(x,\,y))$ is never greater than zero and

$$\begin{aligned}
A_{(z,\,w)}(s,\,(x,\,y)) &= \frac{1}{2} + \lambda \frac{(x-z)(y-w)}{(x-z)^2+(y-w)^2} \\
&\leq \frac{1}{2} + \lambda \frac{\frac{1}{2}\left((x-z)^2+(y-w)^2\right)}{(x-z)^2+(y-w)^2} \\
&= \frac{1}{2}(1+\lambda)
\end{aligned}$$

Fix now an $r_0 > 0$ and take any $r > r_0$. For the pair $(x,\,y)$ for which the supremum of $C_{(z,\,w)}(s,\,(x,\,y))$ is attained when $(x-z)^2+(y-w)^2 = r^2$, we have $x = z + r\cos(\phi_r)$ and $y = w + r\sin(\phi_r)$ for some angle $\phi_r$. Then, we have either $|\cos(\phi_r)| \geq \frac{\sqrt{2}}{2}$ or $|\sin(\phi_r)| \geq \frac{\sqrt{2}}{2}$. If $\cos(\phi_r) \geq \frac{\sqrt{2}}{2}$ holds, we can estimate

$$\begin{aligned}
C_{(z,\,w)}(s,\,(x,\,y)) &= 2r\cos(\phi_r)V^1(z + r\cos(\phi_r)) + 2r\sin(\phi_r)V^2(w + r\sin(\phi_r)) \\
&\leq 2r\cos(\phi_r)V^1(z + r\cos(\phi_r)) \\
&\leq c_1 r
\end{aligned}$$

with $c_1 = \sqrt{2}V^1(z + r_0\frac{\sqrt{2}}{2}) < 0$. In a similar way, by using the fact that both $V^1$ and $V^2$ are strictly decreasing, we can find constants $c_2, c_3, c_4 < 0$ such that $C_{(z,\,w)}(s,\,(x,\,y)) < c_2 r$, $C_{(z,\,w)}(s,\,(x,\,y)) < c_3 r$ and $C_{(z,\,w)}(s,\,(x,\,y)) < c_4 r$, when $\cos(\phi_r) \leq -\frac{\sqrt{2}}{2}$, $\sin(\phi_r) \geq \frac{\sqrt{2}}{2}$ and $\sin(\phi_r) \leq -\frac{\sqrt{2}}{2}$ respectively. Thus, for $c^* = \max\{c_1, c_2, c_3, c_4\} < 0$ we have $C_{(z,\,w)}(s,\,(x,\,y)) < c_* r$, which can be plugged in (A.4) to give the estimate

$$\overline{\beta}(r;(z,\,w),0) \leq \frac{2}{1-\lambda} - 1 + \frac{2c^*}{1+\lambda} r \tag{A.5}$$

for all $r \geq r_0$. This means that for $r_0$ large enough, with the notation of [27] we have

$$\overline{I}_{(z,\,w),r_0}(r) \leq \int_{r_0}^{r} \frac{1}{r'}\left(\frac{2}{1-\lambda} - 1 + \frac{2c^*}{1+\lambda} r'\right) dr'$$

$$\leq \quad c^{**}(r - r_0)$$

for some $c^{**} < 0$ and all $r \geq r_0$. This implies that

$$\int_{r_0}^{+\infty} e^{-\bar{I}_{(z,\,w),r_0}(r)} dr = +\infty$$

and combined with (A.3), it also gives

$$\int_{r_0}^{+\infty} \frac{1}{\underline{\alpha}(r;(z,\,w),0)} e^{\bar{I}_{(z,\,w),r_0}(r)} dr \leq \frac{2}{1-\lambda} \int_{r_0}^{+\infty} e^{c^{**}r} dr < +\infty$$

Therefore, we have that all the assumptions of Theorem 2.5 from [27] are satisfied for $Z = \left(Z^1_\cdot,\, Z^2_\cdot\right)$, which means that $\left(Z^1_\cdot,\, Z^2_\cdot\right)$ is a positive recurrent diffusion, and thus $\left(\sigma^{1,1}_\cdot,\, \sigma^{2,1}_\cdot\right)$ is positive recurrent as well. □

**Proof of Theorem 2.4.** We will show first that each volatility process does never hit zero. Recalling the standard properties of the scale function $S(x)$ of $\sigma^{1,1}_\cdot$ (see [28]), we have that

$$S(x) = \int_{\theta_1}^{x} e^{-\int_{\theta_1}^{y} \frac{2\kappa_1(\theta_1 - z)}{v_1^2 z \tilde{g}^2(z)} dz} dy \tag{A.6}$$

and we need to show that $\lim_{n \to +\infty} S\left(\frac{1}{n}\right) = -\infty$. Since $\sup_{x \in \mathbb{R}} \tilde{g}^2(x) \leq 1 < \frac{2\kappa_1 \theta_1}{v_1^2}$, for $n \geq \frac{1}{\theta_1}$ we have

$$\begin{aligned}
S\left(\frac{1}{n}\right) &= -\int_{\frac{1}{n}}^{\theta_1} e^{\int_{y}^{\theta_1} \frac{2\kappa_1(\theta_1 - z)}{v_1^2 z \tilde{g}^2(z)} dz} dy \\
&\leq -\int_{\frac{1}{n}}^{\theta_1} e^{\int_{y}^{\theta_1} \frac{(\theta_1 - z)}{\theta_1 z} dz} dy \\
&\leq -\int_{\frac{1}{n}}^{\theta_1} e^{\int_{y}^{\theta_1} \frac{1}{z} dz - \int_{y}^{\theta_1} \frac{1}{\theta_1} dz} dy \\
&\leq -\frac{1}{e} \int_{\frac{1}{n}}^{\theta_1} \frac{\theta_1}{y} dy = -\frac{\theta_1}{e}(\ln(n) + \ln(\theta_1))
\end{aligned}$$

where the last tends to $-\infty$ as $n \to +\infty$. This shows that our volatility processes remain positive forever.

Having now that our volatility processes are positive, we can set $Z^i_\cdot = \ln\left(\sigma^{i,1}_\cdot\right)$ for $i \in \{1,\, 2\}$, and we need to show that $\left(Z^1_\cdot,\, Z^2_\cdot\right)$ is a positive recurrent diffusion. We will use the same techniques as in the proof of Theorem 2.4. Again, we can easily determine the infinitesimal generator $L_Z$ of $Z = \left(Z^1_\cdot,\, Z^2_\cdot\right)$, which this time maps any smooth function $F : \mathbb{R}^2 \to \mathbb{R}$ to

$$\begin{aligned}
L_Z F(x,\, y) &= V^1(x) F_x(x,\, y) + V^2(y) F_y(x,\, y) \\
&\quad + \frac{v_1^2 e^{-x} \tilde{g}^2(e^x)}{2} F_{xx}(x,\, y) + \frac{v_2^2 e^{-y} \tilde{g}^2(e^y)}{2} F_{yy}(x,\, y)
\end{aligned}$$

$$+\lambda v_1 v_2 e^{-\frac{x+y}{2}} \tilde{g}(e^y)\tilde{g}(e^y) F_{xy}(x,\, y)$$

for $\lambda = \rho_{2,1}\rho_{2,2} < 1$ and $V^i(x) = e^{-x}\left(\kappa_i\theta_i - \frac{v_i^2}{2}\tilde{g}^2(e^x)\right) - \kappa_i$ for $i \in \{1,\, 2\}$, which are again two continuous and strictly decreasing bijections from $\mathbb{R}$ to itself (this can be shown by using the fact that $\tilde{g}$ is increasing and upper-bounded by 1). Under the notation of [27], by using also the inequality $ab \geq -\frac{a^2+b^2}{2}$ we have

$$\begin{aligned}
A_{(z,\,w)}(s,\,(x,\,y)) &= \frac{1}{2}\left(\frac{v_1^2 e^{-x}\tilde{g}^2(e^x)(x-z)^2}{(x-z)^2+(y-w)^2} + \frac{v_2^2 e^{-y}\tilde{g}^2(e^y)(y-w)^2}{(x-z)^2+(y-w)^2}\right) \\
&\quad +\lambda \frac{v_1 v_2 e^{-\frac{x+y}{2}}\tilde{g}(e^x)\tilde{g}(e^y)(x-z)(y-w)}{(x-z)^2+(y-w)^2} \\
&\geq \frac{1-\lambda}{2}\left(\frac{v_1^2 e^{-x}\tilde{g}^2(e^x)(x-z)^2}{(x-z)^2+(y-w)^2} + \frac{v_2^2 e^{-y}\tilde{g}^2(e^y)(y-w)^2}{(x-z)^2+(y-w)^2}\right) \\
&\geq \frac{(1-\lambda)\min\{v_1^2, v_2^2\}}{2}\min\{e^{-x}\tilde{g}^2(e^x),\, e^{-y}\tilde{g}^2(e^y)\} \qquad \text{(A.7)}
\end{aligned}$$

which is strictly positive. Moreover, we can compute

$$B(s,\,(x,\,y)) = \frac{v_1^2 e^{-x}\tilde{g}^2(e^x)}{2} + \frac{v_2^2 e^{-y}\tilde{g}^2(e^y)}{2} \qquad \text{(A.8)}$$

and

$$C_{(z,\,w)}(s,\,(x,\,y)) = 2\left(V^1(x)(x-z) + V^2(y)(y-w)\right) \qquad \text{(A.9)}$$

for all $(x,\,y),(z,\,w) \in \mathbb{R}^2$. Since the coefficients of $L_Z$, $A_{(z,\,w)}(s,\,(x,\,y))$, $B(s,\,(x,\,y))$ and $C_{(z,\,w)}(s,\,(x,\,y))$ are all continuous, with $A_{(z,\,w)}(s,\,(x,\,y))$ being strictly positive, we can easily verify conditions $A_1$ and $A_5$ from [27]. In order to verify $A_2$, we pick a $N > 0$ and $x, y, \bar{x}, \bar{y} \in [-N,\, N]$, we set

$$M(x,\,y) = \begin{pmatrix} \frac{v_1^2 e^{-x}\tilde{g}^2(e^x)}{2} & \frac{\lambda v_1 v_2 e^{-\frac{x+y}{2}}\tilde{g}(e^x)\tilde{g}(e^y)}{2} \\ \frac{\lambda v_1 v_2 e^{-\frac{x+y}{2}}\tilde{g}(e^x)\tilde{g}(e^y)}{2} & \frac{v_2^2 e^{-y}\tilde{g}^2(e^y)}{2} \end{pmatrix}$$

and we compute

$$\begin{aligned}
\|M(x,\,y) - M(\bar{x},\,\bar{y})\|_{L^2}^2 &= \left(\frac{v_1^2 e^{-x}\tilde{g}^2(e^x)}{2} - \frac{v_1^2 e^{-\bar{x}}\tilde{g}^2(e^{\bar{x}})}{2}\right)^2 \\
&\quad + \left(\frac{v_2^2 e^{-y}\tilde{g}^2(e^y)}{2} - \frac{v_2^2 e^{-\bar{y}}\tilde{g}^2(e^{\bar{y}})}{2}\right)^2 \\
&\quad + \left(\frac{\lambda v_1 v_2 e^{-\frac{x+y}{2}}\tilde{g}(e^x)\tilde{g}(e^y)}{2} - \frac{\lambda v_1 v_2 e^{-\frac{\bar{x}+\bar{y}}{2}}\tilde{g}(e^{\bar{x}})\tilde{g}(e^{\bar{y}})}{2}\right)^2 \\
&\leq C_N \left\|(x,\,y) - (\bar{x},\,\bar{y})\right\|_{L^2(\mathbb{R}^2)}^2 \qquad \text{(A.10)}
\end{aligned}$$

where we have used the two-dimensional Mean Value Theorem on each of the three terms, and the fact all the involved functions have a bounded gradient in $[-N,\, N]^2$ (since $\tilde{g}$ has continuous derivatives). Thus, for $\delta_N(r) = C_N r$, we obtain $A_2$ as well. Next, for some $r_0 > 0$ and all $r \geq r_0$, we compute

$$\underline{\alpha}(r; (z,\, w), 0) = \inf_{(x-z)^2+(y-w)^2=r^2} A_{(z,\,w)}(s,\,(x,\,y))$$

$$\geq \quad \frac{(1-\lambda)\min\{v_1^2, v_2^2\}}{2} e^{-\max\{z,\,w\}-r}\tilde{g}^2(e^{\max\{z,\,w\}+r}) \tag{A.11}$$

and

$$\begin{aligned}
\overline{\beta}(r;(z,\,w),0) &= \sup_{(x-z)^2+(y-w)^2=r^2} \frac{B(s,\,(x,\,y)) - A_{(z,\,w)}(s,\,(x,\,y)) + C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))} \\
&= -1 + \sup_{(x-z)^2+(y-w)^2=r^2} \frac{B(s,\,(x,\,y)) + C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))}
\end{aligned} \tag{A.12}$$

where again we choose $z$ and $w$ to be the unique roots of $V^1(x)$ and $V^2(y)$ respectively. Then, by setting $x = z + r\cos(\phi_r)$ and $y = w + r\sin(\phi_r)$ with $\phi_r \in [0,\,2\pi]$ for the $(x,\,y)$ for which the above supremum is attained, since $\tilde{g}$ is increasing we have

$$\begin{aligned}
C_{(z,\,w)}(s,\,(x,\,y)) &= 2\left(e^{-x}\left(\kappa_1\theta_1 - \frac{v_1^2}{2}\tilde{g}^2(e^x)\right) - \kappa_1\right)(x-z) \\
&\quad +2\left(e^{-y}\left(\kappa_2\theta_2 - \frac{v_2^2}{2}\tilde{g}^2(e^y)\right) - \kappa_2\right)(y-w) \\
&\leq 2\left(e^{-x}\left(\kappa_1\theta_1 - \frac{v_1^2}{2}\tilde{g}^2(e^z)\right) - \kappa_1\right)(x-z) \\
&\quad +2\left(e^{-y}\left(\kappa_2\theta_2 - \frac{v_2^2}{2}\tilde{g}^2(e^w)\right) - \kappa_2\right)(y-w) \\
&= 2\kappa_1\left(e^{z-x}-1\right)(x-z) + 2\kappa_2\left(e^{w-y}-1\right)(y-w) \\
&\leq 2\min\{\kappa_1,\kappa_2\}\left(\left(e^{z-x}-1\right)(x-z) + \left(e^{w-y}-1\right)(y-w)\right) \\
&= \kappa r\left(\left(e^{-r\cos(\phi_r)}-1\right)\cos(\phi_r) + \left(e^{-r\sin(\phi_r)}-1\right)\sin(\phi_r)\right)
\end{aligned} \tag{A.13}$$

for $\kappa = 2\min\{\kappa_1,\kappa_2\}$, and since $\tilde{g}$ is bounded, for $\xi = \max\left\{\frac{v_1^2}{2},\,\frac{v_2^2}{2}\right\}\sup_{x\in\mathbb{R}}\tilde{g}(x)$ we can also show that

$$\begin{aligned}
A_{(z,\,w)}(s,\,(x,\,y)) &\leq \xi\left(e^{-x}\frac{(x-z)^2}{(x-z)^2+(y-w)^2} + e^{-y}\frac{(y-w)^2}{(x-z)^2+(y-w)^2}\right) \\
&= \xi\left(e^{-z-r\cos(\phi_r)}\cos^2(\phi_r) + e^{-w-r\sin(\phi_r)}\sin^2(\phi_r)\right) \\
&= \xi\left(e^{-z}\left(e^{-r\cos(\phi_r)}-1\right)\cos^2(\phi_r) + e^{-w}\left(e^{-r\sin(\phi_r)}-1\right)\sin^2(\phi_r)\right) \\
&\quad +\xi\left(e^{-z}\cos^2(\phi_r) + e^{-w}\sin^2(\phi_r)\right) \\
&\leq -\xi\left(e^{-z}\left(e^{-r\cos(\phi_r)}-1\right)\cos(\phi_r) + e^{-w}\left(e^{-r\sin(\phi_r)}-1\right)\sin(\phi_r)\right) \\
&\quad +\xi\left(e^{-z}+e^{-w}\right)
\end{aligned} \tag{A.14}$$

where we have also used the elementary inequality $(e^{ab}-1)a^2 \leq -(e^{ab}-1)a$ for $|a| \leq 1$ and $b < 0$. By using (A.13) and (A.14) now we obtain

$$\frac{C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))} \leq -r\frac{\kappa}{\xi}\frac{\ell(r)}{\ell(r)+\xi\left(e^{-z}+e^{-w}\right)} \tag{A.15}$$

where

$$
\begin{aligned}
\ell(r) &= -\xi\left(e^{-z}\left(e^{-r\cos(\phi_r)}-1\right)\cos(\phi_r)+e^{-w}\left(e^{-r\sin(\phi_r)}-1\right)\sin(\phi_r)\right)\\
&\geq -\xi e^{-z}\left(e^{-r\cos(\phi_r)}-1\right)\cos(\phi_r)\\
&= \xi e^{-z}\left|e^{-r\cos(\phi_r)}-1\right||\cos(\phi_r)|\\
&\geq \xi\min\{e^{-z},e^{-w}\}\frac{\sqrt{2}}{2}\min\left\{\left|e^{-r_0\frac{\sqrt{2}}{2}}-1\right|,\left|e^{r_0\frac{\sqrt{2}}{2}}-1\right|\right\} \qquad (A.16)
\end{aligned}
$$

since we take $r \geq r_0$, and without loss of generality we can assume that $|\cos(\phi_r)| \geq \frac{\sqrt{2}}{2}$. Thus, (A.15) implies that there is a universal $c^* < 0$ such that

$$
\frac{C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))} \leq c^* r \qquad (A.17)
$$

when $r \geq r_0$. Plugging the last in (A.12) we obtain

$$
\begin{aligned}
\overline{\beta}(r;(z,\,w),0) &= -1+\sup_{(x-z)^2+(y-w)^2=r^2}\frac{B(s,\,(x,\,y))+C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))}\\
&\leq -1+pc^*r+\sup_{(x-z)^2+(y-w)^2=r^2}\frac{B(s,\,(x,\,y))+(1-p)C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))}
\end{aligned}
\qquad (A.18)
$$

for all $r \geq r_0$ and a $p \in [0,\,1]$ which will be chosen later. We will show now that the last term in the RHS of the above is negative for $r_0$ large enough (depending on $p$). Indeed, by using (A.13), the definition of $B(s,\,(x,\,y))$, and the fact that $\tilde{g}$ is upper-bounded, we can obtain the estimate

$$
\begin{aligned}
&\sup_{(x-z)^2+(y-w)^2=r^2}\frac{B(s,\,(x,\,y))+(1-p)C_{(z,\,w)}(s,\,(x,\,y))}{A_{(z,\,w)}(s,\,(x,\,y))}\\
&\leq \sup_{(x-z)^2+(y-w)^2=r^2}\frac{\kappa^*\left(\left(e^{z-x}-1\right)(x-z)+\left(e^{w-y}-1\right)(y-w)\right)+\xi(e^{-x}+e^{-y})}{A_{(z,\,w)}(s,\,(x,\,y))}
\end{aligned}
\qquad (A.19)
$$

where as before, we have $\xi = \max\left\{\frac{v_1^2}{2},\,\frac{v_2^2}{2}\right\}\sup_{x\in\mathbb{R}}\tilde{g}(x)$, and $\kappa^* = (1-p)\kappa$. The numerator of the last quantity can be easily be show to tend to $-\infty$ when $x$ or $y$ tends to $\pm\infty$, which happens when $r \to +\infty$. Thus, for $r \geq r_0$ with $r_0$ large enough, the RHS of (A.19) is negative, which can be plugged in (A.17) to give

$$
\overline{\beta}(r;(z,\,w),0) \quad \leq \quad -1+pc^*r
$$

for all $r \geq r_0$, with $c^* < 0$. This means that as in the previous case, with the notation of [27], for $r_0$ large enough we have

$$
\overline{I}_{(z,\,w),r_0}(r) \quad \leq \quad \int_{r_0}^{r}\frac{1}{r'}\left(-1+pc^*r'\right)dr' \leq pc^*(r-r_0) \qquad (A.20)
$$

with $c^* < 0$, for all $r \geq r_0$. This implies that

$$\int_{r_0}^{+\infty} e^{-\overline{I}_{(z,\,w),r_0}(r)} dr = +\infty$$

which means that our two-dimensional process is recurrent (see Theorem 2.4 in [27]). However, what we need is positive recurrence, and thus we need to show that

$$\int_{r_0}^{+\infty} \frac{1}{\underline{\alpha}(r;(z,\,w),0)} e^{\overline{I}_{(z,\,w),r_0}(r)} dr < +\infty.$$

By using (A.20), (A.11) and the fact that $\tilde{g}$ is lower-bounded by something positive, we can bound the last integral by a positive multiple of

$$\int_{r_0}^{+\infty} e^{(1+pc^*)r} dr$$

which is finite when $pc^* < -1$. Taking $\eta$ small enough and $r_0$ big enough, we can force $e^{-w}$ and $e^{-z}$ to be arbitrarily close to each other, and the lower bound of $\ell$ (given by (A.16)) to be arbitrarily close to $\frac{\sqrt{2}\xi}{2}e^{-z}$. This makes $c^*$ (obtained in (A.15)) arbitrarily close to $-\frac{\kappa}{\xi}\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}+2} < -1$ by the assumptions of the Theorem and the definitions of $\kappa$ and $\xi$. Thus, if $p$ is chosen to be very close to 1, we can achieve $pc^* < -1$ as well, which gives the desired result. □

**Proof of Lemma 5.1.** First, we will show that each volatility process has a finite $2p$-moment for any $p \in \mathbb{N}$. Indeed, we fix a $p \in \mathbb{N}$ and we consider the sequence of stopping times $\{\tau_{n,\epsilon} : n \in \mathbb{N}\}$, where $\tau_{n,\epsilon} = \inf\{t \geq 0 : \sigma_t^{i,\epsilon} > n\}$. Setting $\sigma_t^{i,n,\epsilon} = \sigma_{t\wedge\tau_{n,\epsilon}}^{i,\epsilon}$, by Ito's formula we have

$$\begin{aligned}
\left(\sigma_t^{i,n,\epsilon} - \theta_i\right)^{2p} &= \left(\sigma_0^{i,n,\epsilon} - \theta_i\right)^{2p} - \frac{2p\kappa_i}{\epsilon}\int_0^t \mathbb{I}_{[0,\tau_{n,\epsilon}]}(s)\left(\sigma_s^{i,n,\epsilon} - \theta_i\right)^{2p} ds \\
&\quad + 2p\xi_i \int_0^t \mathbb{I}_{[0,\tau_{n,\epsilon}]}(s)\left(\sigma_s^{i,n,\epsilon} - \theta_i\right)^{2p-1} g\left(\sigma_s^{i,n,\epsilon}\right) d\tilde{B}_s^i \\
&\quad + p(2p-1)\xi_i^2 \int_0^t \mathbb{I}_{[0,\tau_{n,\epsilon}]}(s)\left(\sigma_s^{i,n,\epsilon} - \theta_i\right)^{2p-2} g^2\left(\sigma_s^{i,n,\epsilon}\right) ds
\end{aligned} \tag{A.21}$$

for $\tilde{B}_s^i = \sqrt{1-\rho_{2,i}^2}B_t^i + \rho_{2,i}B_t^0$, where the stochastic integral is a Martingale. Taking expectations, setting $f(t,n,p,\epsilon) = \mathbb{E}\left[\left(\sigma_t^{i,n,\epsilon} - \theta_i\right)^{2p}\right]$ and using the growth condition of $g$ and simple inequalities, we can easily obtain

$$f(t,n,p,\epsilon) \leq M + M' \int_0^t f(s,n,p,\epsilon) ds$$

with $M, M'$ depending only on $p$, $c_g$ and the bounds of $\sigma^i, \xi_i, \theta_i$. Thus, using Gronwall's inequality we get a uniform (in $n$) estimate for $f(t,n,p,\epsilon)$, and then by Fatou's lemma we obtain the desired finiteness of $f(t,p,\epsilon) := \mathbb{E}\left[\left(\sigma_t^{i,\epsilon} - \theta_i\right)^{2p}\right]$. This implies the almost sure finiteness of $f_{\mathcal{C}}(t,p,\epsilon) := \mathbb{E}\left[\left(\sigma_t^{i,\epsilon} - \theta_i\right)^{2p} | \mathcal{C}\right]$ as well. Taking then expectations given $\mathcal{C}$ and $n \to +\infty$ on (A.21), and using the Monotone Convergence Theorem (all quantities are monotone for large enough $n$) and the growth condition on $g$, we find that

$$f_{\mathcal{C}}(t,p,\epsilon) \leq M + \left(M' - \frac{2\kappa_i}{\epsilon}\right) \int_0^t f_{\mathcal{C}}(s,p,\epsilon)ds$$

where again, $M, M'$ depend only on $p$, $c_g$ and the bounds of $\sigma^i, \xi_i, \theta_i$. Using Grownwall's inequality again on the above, we finally have

$$\int_0^t f_{\mathcal{C}}(s,p,\epsilon)ds \leq M \int_0^t e^{\left(M' - \frac{2\kappa_i}{\epsilon}\right)(t-s)} ds \tag{A.22}$$

where the integral on the RHS of the above is bounded and tends to 0 as $\epsilon \to 0^+$. Thus, by the Dominated Convergence Theorem we have that $\int_0^t f(s,p,\epsilon)ds = \mathbb{E}\left[\int_0^t f_{\mathcal{C}}(s,p,\epsilon)ds\right]$ tends also to zero, and this gives the desired convergence result. □

**Proof of Lemma 5.2.** From the hypothesis, we can write $Z_t = Z_0 + \int_0^t z_s ds$ with $Z_.$, $z_.$ bounded in $[0,\, t]$ by $M_{z,t}$. For any $j > 0$, since $g$ has a linear growth (in the worst case), there exist some $a_j, b_j > 0$ such that $g^{2j}(z) \leq a_j(z - \theta_1)^{2j} + b_j$ for all $z \in \mathbb{R}$. Then, for a given sequence $\epsilon_n \to 0^+$, by using Ito's formula we have

$$\begin{aligned}
&-\frac{\kappa_1}{\epsilon_n}\int_0^t \left(h\left(\sigma_s^{1,\epsilon_n}\right) - h\left(\theta_1\right)\right)^2 Z_s ds \\
&\quad = -\frac{\kappa_1}{\epsilon_n}\int_0^t \left(\sum_{m=1}^{+\infty} \frac{1}{m!} h^{(m)}(\theta_1)\left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^m\right)^2 Z_s ds \\
&\quad = -\frac{\kappa_1}{\epsilon_n}\int_0^t \sum_{m_1,m_2=1}^{+\infty} \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!}\left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2} Z_s ds \\
&\quad = -\frac{\kappa_1}{\epsilon_n}\int_0^t \sum_{m_1,m_2=1}^{+\infty} \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!}\int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2} Z_s ds \\
&\quad = \sum_{m_1,m_2=1}^{+\infty} \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!(m_1+m_2)}\left(\sigma_t^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2} Z_t \\
&\quad\quad - \sum_{m_1,m_2=1}^{+\infty} \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!(m_1+m_2)}\left(\sigma^1 - \theta_1\right)^{m_1+m_2} Z_0 \\
&\quad\quad - \sum_{m_1,m_2=1}^{+\infty} \xi_1 \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!}\int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2-1} g\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1 \\
&\quad\quad - \sum_{m_1,m_2=1}^{+\infty} \frac{\xi_1^2}{2}\frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)(m_1+m_2-1)}{m_1!m_2!} \\
&\quad\quad\quad\quad \times \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2-2} g^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds
\end{aligned}$$

$$
- \sum_{m_1,m_2=1}^{+\infty} \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!(m_1+m_2)} \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2} z_s ds. \tag{A.23}
$$

Observe now that

$$
\begin{aligned}
&\sum_{m_1,m_2=1}^{+\infty} \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!(m_1+m_2)} \left(\sigma^1 - \theta_1\right)^{m_1+m_2} Z_0 \\
&= \sum_{m_1,m_2=1}^{+\infty} \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!} \int_{\theta_1}^{\sigma^1} \left(y - \theta_1\right)^{m_1+m_2-1} dy Z_0 \\
&= \int_{\theta_1}^{\sigma^1} \frac{1}{(y-\theta_1)} \sum_{m_1,m_2=1}^{+\infty} \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)}{m_1!m_2!} \left(y - \theta_1\right)^{m_1+m_2} dy Z_0 \\
&= \int_{\theta_1}^{\sigma^1} \frac{1}{(y-\theta_1)} \left( \sum_{m_1=1}^{+\infty} \frac{h^{(m_1)}(\theta_1)}{m_1!} \left(y - \theta_1\right)^{m_1} \right)^2 dy Z_0 \\
&= Z_0 \int_{\theta_1}^{\sigma^1} \frac{1}{(y-\theta_1)} \left(h\left(y\right) - h\left(\theta_1\right)\right)^2 dy
\end{aligned} \tag{A.24}
$$

while for $m_1 = m_2 = 1$ we have

$$
\begin{aligned}
&-\frac{\xi_1^2}{2} \frac{h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)(m_1+m_2-1)}{m_1!m_2!} \\
&\qquad \times \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m_1+m_2-2} g^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds \\
&\qquad = -\frac{\xi_1^2}{2} \left(h'\left(\theta_1\right)\right)^2 \int_0^t g\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds.
\end{aligned} \tag{A.25}
$$

which converges to $-\frac{\xi_1^2}{2}\left(h'\left(\theta_1\right) g^2\left(\theta_1\right)\right)^2 \int_0^t Z_s ds$ in $L^2$ for all $t \geq 0$, as $n \to +\infty$. To show this convergence, we compute the squared $L^2$ norm of the difference between the sequence and the desired limit, which equals

$$
\begin{aligned}
&\mathbb{E}\left[\frac{\xi_1^4}{4}\left(h'\left(\theta_1\right)\right)^4 \left(\int_0^t \left(g^2\left(\sigma_s^{1,\epsilon_n}\right) - g^2\left(\theta_1\right)\right) Z_s ds\right)^2\right] \\
&\qquad \leq \frac{\|\xi_1\|_\infty^4}{4} M_h^4 t M_{z,t}^2 \mathbb{E}\left[\int_0^t \left(g^2\left(\sigma_s^{1,\epsilon_n}\right) - g^2\left(\theta_1\right)\right)^2 ds\right],
\end{aligned}
$$

and since the derivative of $g^2$ is bounded by some $M_g > 0$, by using the mean value theorem we have

$$
\mathbb{E}\left[\int_0^t \left(g^2\left(\sigma_s^{1,\epsilon_n}\right) - g^2\left(\theta_1\right)\right)^2 ds\right] \quad \leq \quad M_g^2 \mathbb{E}\left[\int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^2 ds\right]
$$

which tends to zero as $n \to +\infty$ (by Lemma 5.1).

We denote now by $S_n$ the sum of all the other terms in (A.23), so next we need to show that $\|S_n\|_{L^2(\Omega)}$ tends to zero in a subsequence $\{\epsilon_{k_n} : n \in \mathbb{N}\}$ of $\{\epsilon_n : n \in \mathbb{N}\}$, for almost all $t \geq 0$ as $n \to +\infty$. Writing $\mathbb{E}_{\mathcal{C}}$ for the expectation given the coefficients and $L_{\mathcal{C}}^2$

for the corresponding $L^2$ norm, we obviously have $\|S_n\|_{L^2(\Omega)} = \sqrt{\mathbb{E}\left[\|S_n\|^2_{L^2_{\mathcal{C}}(\Omega)}\right]}$, and by the triangle inequality we have that $\|S_n\|_{L^2_{\mathcal{C}}(\Omega)}$ is less than the sum of the $L^2_{\mathcal{C}}(\Omega)$ norms of the terms of $S_n$, which equals

$$\begin{aligned}
&\sum_{m_1,m_2=1}^{+\infty} \frac{\left|h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)\right|}{m_1!m_2!(m_1+m_2)} \sqrt{\mathbb{E}\left[\left(\sigma_t^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2)} Z_t^2 \,|\, \mathcal{C}\right]} \\
&\quad + \sum_{m_1,m_2=1}^{+\infty} \xi_1 \frac{\left|h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)\right|}{m_1!m_2!} \\
&\qquad \times \sqrt{\mathbb{E}\left[\left(\int_0^t \left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2-1} g\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1\right)^2 \,|\, \mathcal{C}\right]} \\
&\quad + \sum_{m_1+m_2\geq 3}^{+\infty} \frac{\xi_1^2}{2} \frac{\left|h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)(m_1+m_2-1)\right|}{m_1!m_2!} \\
&\qquad \times \sqrt{\mathbb{E}\left[\left(\int_0^t \left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2-2} g^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds\right)^2 \,|\, \mathcal{C}\right]} \\
&\quad + \sum_{m_1,m_2=1}^{+\infty} \frac{\left|h^{(m_1)}(\theta_1)h^{(m_2)}(\theta_1)\right|}{m_1!m_2!(m_1+m_2)} \sqrt{\mathbb{E}\left[\left(\int_0^t \left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m_1+m_2} z_s ds\right)^2 \,|\, \mathcal{C}\right]}.
\end{aligned} \tag{A.26}$$

Then, by using Ito's isometry, the boundedness of $Z.$ and $z.$ in $[0,\, t]$ by $M_{z,t}$, the linear growth of $g$ combined with the triangle inequality, the boundedness of the sequence $\left\{h^{(n)}(\theta_1) : n \in \mathbb{N}\right\}$ by the deterministic constant $M_h$, and finally the Cauchy-Schwartz inequality, we can bound the above by a multiple of

$$\begin{aligned}
&\sum_{m_1,m_2=1}^{+\infty} \frac{1}{m_1!m_2!(m_1+m_2)} \sqrt{\mathbb{E}\left[\left(\sigma_t^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2)} \,|\, \mathcal{C}\right]} \\
&\quad + \sum_{m_1,m_2=1}^{+\infty} \frac{1}{m_1!m_2!} \sqrt{\int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2)} \,|\, \mathcal{C}\right] ds} \\
&\quad + \sum_{m_1,m_2=1}^{+\infty} \frac{1}{m_1!m_2!} \sqrt{\int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2-1)} \,|\, \mathcal{C}\right] ds} \\
&\quad + \sum_{m_1+m_2\geq 3}^{+\infty} \frac{(m_1+m_2-1)}{m_1!m_2!} \sqrt{\int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2-1)} \,|\, \mathcal{C}\right] ds} \\
&\quad + \sum_{m_1+m_2\geq 3}^{+\infty} \frac{(m_1+m_2-1)}{m_1!m_2!} \sqrt{\int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2-2)} \,|\, \mathcal{C}\right] ds} \\
&\quad + \sum_{m_1,m_2=1}^{+\infty} \frac{1}{m_1!m_2!(m_1+m_2)} \sqrt{\int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{2(m_1+m_2)} \,|\, \mathcal{C}\right] ds}.
\end{aligned} \tag{A.27}$$

Using now the inequality $x < \frac{1}{\eta^2}x^2 + \eta^2$ to get rid of the square roots, then Tonelli's Theorem to interchange the sums with the integrals and the expectations, and finally a few trivial inequalities like $m_1 + m_2 - 1 \leq m_1 m_2$, we find that each of the above six sums is less than

$$\mathbb{E}\left[\sum_{m_1+m_2\geq 1}^{+\infty} \frac{\frac{1}{\eta^2}\left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{2(m_1+m_2)} + \eta^2}{m_1! m_2!} \,|\, \mathcal{C}\right]$$

$$= \frac{2}{\eta^2}\mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^2 \,|\, \mathcal{C}\right] + \frac{1}{\eta^2}\mathbb{E}\left(\left[\sum_{m=1}^{+\infty} \frac{\left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{2m}}{m!}\right)^2 \,|\, \mathcal{C}\right]$$

$$+2\eta^2 + \eta^2\left(\sum_{m=1}^{+\infty}\frac{1}{m!}\right)^2$$

$$= \frac{1}{\eta^2}\left(2\mathbb{E}\left[\left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^2 \,|\, \mathcal{C}\right] + \mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^2} - 1\right)^2 \,|\, \mathcal{C}\right]\right)$$

$$+\eta^2\left(2 + (e-1)^2\right)$$

computed at $s = t$, or its integral for $s \in [0, t]$. In both cases, the desired result follows because in a subsequence $\{\epsilon_{k_n}\}$ of $\{\epsilon_n\}$, for very small $\eta > 0$ and very large $n$, the $L^2(\Omega)$ norm of both quantities can be made arbitrarily small, for almost all $t > 0$. Indeed, for any $t > 0$ we have

$$\sqrt{\mathbb{E}\left[\left(\int_0^t \left(2\mathbb{E}\left[\left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 \,|\, \mathcal{C}\right] + \mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon} - \theta_1\right)^2} - 1\right)^2 \,|\, \mathcal{C}\right]\right) ds\right)^2\right]}$$

$$\leq 2\sqrt{\mathbb{E}\left[\left(\int_0^t \left(\mathbb{E}\left[\left(\sigma_s^{1,\epsilon} - \theta_1\right)^2 \,|\, \mathcal{C}\right]\right) ds\right)^2\right]}$$

$$+\sqrt{\mathbb{E}\left[\left(\int_0^t \left(\mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon} - \theta_1\right)^2} - 1\right)^2 \,|\, \mathcal{C}\right]\right) ds\right)^2\right]}$$

$$\leq 2\sqrt{\int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon} - \theta_1\right)^4\right] ds} + \sqrt{\int_0^t \mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon} - \theta_1\right)^2} - 1\right)^4\right] ds} \tag{A.28}$$

and in a similar way

$$\sqrt{\mathbb{E}\left[\left(2\mathbb{E}\left[\left(\sigma_t^{1,\epsilon} - \theta_1\right)^2 \,|\, \mathcal{C}\right] + \mathbb{E}\left[\left(e^{\left(\sigma_t^{1,\epsilon} - \theta_1\right)^2} - 1\right)^2 \,|\, \mathcal{C}\right]\right)^2\right]}$$

$$\leq 2\sqrt{\mathbb{E}\left[\left(\sigma_t^{1,\epsilon} - \theta_1\right)^4\right]} + \sqrt{\mathbb{E}\left[\left(e^{\left(\sigma_t^{1,\epsilon} - \theta_1\right)^2} - 1\right)^4\right]} \tag{A.29}$$

and we will show that the RHS of (A.28) tends to zero as $\epsilon \to 0^+$ for all $t > 0$, which implies also that the RHS of (A.29) tends to zero in subsequences for almost all $t > 0$. For the first term of the RHS of (A.28), convergence to zero follows by Lemma 5.1, while for second term we can use the inequality $e^x - 1 \le xe^x$ for $x \ge 0$ to obtain

$$\begin{aligned}\int_0^t \mathbb{E}\left[\left(e^{\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}-1\right)^4\right] ds &\le \int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon}-\theta_1\right)^4 e^{4\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}\right] ds \\ &\le \sqrt{\int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon}-\theta_1\right)^8\right] ds}\sqrt{\int_0^t \mathbb{E}\left[e^{8\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}\right] ds}\end{aligned}$$

with $\int_0^t \mathbb{E}\left[\left(\sigma_s^{1,\epsilon}-\theta_1\right)^8\right] ds$ tending to zero as $\epsilon \to 0^+$, and $\int_0^t \mathbb{E}\left[e^{8\left(\sigma_s^{1,\epsilon}-\theta_1\right)^2}\right] ds \le 1$ for small enough $\epsilon$. The last is true because

$$\begin{aligned}\left(\sigma_t^{1,\epsilon}-\theta_1\right)^2 &= \left(\sigma_t^{1,\epsilon}-\theta_1\right)^2 - \frac{2\kappa_1}{\epsilon}\int_0^t \left(\sigma_t^{1,\epsilon}-\theta_1\right)^2 ds \\ &\quad +2\xi_1 \int_0^t \left(\sigma_t^{1,\epsilon}-\theta_1\right) g\left(\sigma_t^{1,\epsilon}\right) d\tilde{B}_s^1 \\ &\quad +\xi_1^2 \int_0^t g^2\left(\sigma_t^{1,\epsilon}\right) ds \end{aligned} \tag{A.30}$$

which means that when $\epsilon < \epsilon_0$ for $\epsilon_0$ small enough, we can use the boundedness of $g, \kappa_1$ and $\xi_1$ to bound $e^{8\left(\sigma_t^{1,\epsilon}-\theta_1\right)^2}$ by the Doleans exponential of $4\xi_1 \int_0^t \left(\sigma_t^{1,\epsilon}-\theta_1\right) g\left(\sigma_t^{1,\epsilon}\right) d\tilde{B}_s^1$, which is a local Martingale with expectation less than 1. Observe that $\epsilon_0$ should not depend on the coefficients, and this is why a deterministic positive lower bound for $\kappa_1$ is needed. This completes the proof of the first convergence result.

If $h$ is a polynomial, there are only finitely many terms in (A.26), so we can replace the infinite sums in (A.27) with finite ones. In that case, we can just use the boundedness of the coefficients and 5.1 to show that the finitely many terms in these sums tend all to zero in $L^2(\Omega)$, without the need to assume that $g$ is bounded.

The second convergence result can be obtained in the same way, since we can compute

$$\begin{aligned}&-\frac{\kappa_1}{\epsilon_n}\int_0^t \left(h\left(\sigma_s^{1,\epsilon_n}\right) - h(\theta_1)\right) Z_s ds \\ &\quad = -\frac{\kappa_1}{\epsilon_n}\int_0^t \sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!}\left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^m Z_s ds \\ &\quad = \sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!m}\left(\sigma_t^{1,\epsilon_n}-\theta_1\right)^m Z_t \\ &\qquad - \sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!m}\left(\sigma^1-\theta_1\right)^m Z_0 \\ &\qquad - \sum_{m=1}^{+\infty} \xi_1 \frac{h^{(m)}(\theta_1)}{m!}\int_0^t \left(\sigma_s^{1,\epsilon_n}-\theta_1\right)^{m-1} g\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1 \\ &\qquad - \sum_{m=1}^{+\infty} \frac{\xi_1^2}{2}\frac{h^{(m)}(\theta_1)(m-1)}{m!}\end{aligned}$$

$$\times \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m-2} g^2 \left(\sigma_s^{1,\epsilon_n}\right) Z_s ds$$
$$- \sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!m} \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^m z_s ds. \tag{A.31}$$

where

$$\begin{aligned} \sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!m} \left(\sigma^1 - \theta_1\right)^m Z_0 &= \int_{\theta_1}^{\sigma^1} \frac{1}{y-\theta_1} \sum_{m=1}^{+\infty} \frac{h^{(m)}(\theta_1)}{m!} (y-\theta_1)^m \, dy Z_0 \\ &= Z_0 \int_{\theta_1}^{\sigma^1} \frac{h(y) - h(\theta_1)}{y - \theta_1} dy \end{aligned}$$

while for $m = 1$ and $m = 2$ we have

$$\begin{aligned} &\xi_1 \frac{h^{(m)}(\theta_1)}{m!} \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m-1} g\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1 \\ &\qquad = \xi_1 h'(\theta_1) \int_0^t g\left(\sigma_s^{1,\epsilon_n}\right) Z_s d\tilde{B}_s^1 \end{aligned}$$

and

$$\begin{aligned} &\frac{\xi_1^2}{2} \frac{h^{(m)}(\theta_1)(m-1)}{m!} \int_0^t \left(\sigma_s^{1,\epsilon_n} - \theta_1\right)^{m-2} g^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds \\ &\qquad = \frac{\xi_1^2}{2} h''(\theta_1) \int_0^t g^2\left(\sigma_s^{1,\epsilon_n}\right) Z_s ds \end{aligned} \tag{A.32}$$

respectively, whose $L^2$ distances from $\xi_1 h'(\theta_1) g(\theta_1) \int_0^t Z_s d\tilde{B}_s^1$ and $\frac{\xi_1^2}{4} h''(\theta_1) g^2(\theta_1) \int_0^t Z_s ds$ respectively are bounded by multiples of $\mathbb{E}\left[\int_0^t \left(g^2\left(\sigma_s^{1,\epsilon_n}\right) - g^2(\theta_1)\right)^2 ds\right]$ which tends to zero, as we have shown earlier. To show that the $L^2$ norm of the sum of all the other terms tends to zero, we need to follow the same steps we followed for showing the corresponding result in the proof of the first convergence result. The only difference is that this time the computations involve controlling certain sums by exponential Taylor series, and not by the squares of these series. The proof of the Lemma is now complete. □

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